Cyclic Pursuit for Spacecraft Formation Control

Jaime L. Ramirez, Marco Pavone, Emilio Frazzoli

Abstract— **In this paper we present decentralized control policies for spacecraft formation, based on the simple idea of cyclic pursuit. First, we extend the existing cyclic-pursuit control laws devised for a single integrator to a double integrator. Such dynamically extended control laws only require relative measurements of position and velocity of two leading neighbors, and allow spacecraft to converge, globally in three dimensions, to a single point, a circle or a logarithmic spiral pattern, depending on the value of a control parameter. Second, we modify the aforementioned control laws to allow convergence to elliptical trajectories in three dimensions, that represent natural trajectories of the relative motion between spacecraft in a rotating frame around the earth. Finally, we discuss the applicability of the devised control laws to the problem of spacecraft coordination for interferometric imaging, and their implementation on a 4DOF testbed.**

I. Introduction

In recent years, the idea of *distributing* the functionalities of complex agents among simple and cooperative agents is attracting an ever increasing interest. In fact, decentralized (i.e., without a leader) multi-agent systems present several advantages. For instance, the intrinsic parallelism of a decentralized multi-agent system provides robustness to failures of single agents, and in many cases can guarantee better time efficiency. Moreover, it is possible to reduce the total implementation and operation cost, increase reactivity and system reliability, and add flexibility and modularity to monolithic approaches. Potential applications of multi-agent systems include search and rescue operations, humanitarian demining, manipulation in hazardous environments, planetary exploration, environmental monitoring and surveillance.

In this context, geometric pattern formation problems are of particular interest, especially for their connection to formation flight. Indeed, the innovative idea of *distributing* the functionality of large satellites among smaller, cooperative satellites flying in formation has been considered for numerous space missions. For instance, a cluster of satellites flying in formation for high-resolution, synthetic-aperture imaging can act as a sparse aperture with an effective dimension larger than the one that can be achieved by a single, larger satellite [2].

Recently, many theoretical control strategies have been proposed to let multi-agent system achieve a geometric pattern just relying on simple *local* rules. Justh and Krishnaprasad [18] presented a differential geometric setting for the problem of formation control and proposed two strategies to achieve, respectively, rectilinear and circle formation; their approach, however, requires all-to-all communication among agents. Jadbabaie *et al.* [19] formally proved that the nearest neighbor algorithm by Vicsek [20] causes all agents to eventually move in the same direction, despite the absence of centralized coordination and despite the fact that each agent's set of nearest neighbors changes with time as the system evolves. Olfati-Saber and Murray [23] and Leonard *et al.* [24] used potential function theory to prescribe flocking behavior. Lin *et al.* [25] exploited cyclic pursuit to achieve alignment among agents, while Marshall *et al.* in [26] and in [27] extended the classic cyclic pursuit to a system of wheeled vehicles, each subject to a single non-holonomic constraint, and studied the possible equilibrium formations and their stability.

In [15], we developed distributed control policies that allow the agents to achieve different symmetric formations. The key features of our approach are global stability and the possibility to achieve with the same simple control law different formations, namely rendezvous to a single point, circles or logarithmic spirals in two dimensions. Our approach generalizes the classic cyclic pursuit strategy (where each agent i pursues the next $i+1$, modulo n) by letting each of the n mobile agents, modeled as single integrators, pursue its leading neighbor along the line of sight *rotated* by a common offset angle α. Such approach is attractive since it is decentralized and requires the minimum number of communication links (n links for n agents) to achieve a formation.

Jaime Ramirez is a PhD Candidate of the Department of Aeronautics and Astronautics at the Massachusetts Institute of Technology ramirez@mit.edu

Marco Pavone is a PhD Candidate of the Department of Aeronautics and Astronautics at the Massachusetts Institute of Technology pavone@mit.edu

Emilio Frazzoli is an Associate professor of the Department of Aeronautics and Astronautics at the Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, MA. 02139 frazzoli@mit.edu

The contribution of this paper is threefold. First, we extend the generalized cyclic pursuit control laws introduced in [15] to double integrators. The resulting control laws allow the agents to converge globally (i.e., from any set of initial conditions), in three dimensions, to a single point, a circle or a logarithmic spiral pattern, depending on the value of a control parameter; remarkably, they only require *relative* measurements of position and velocity of two leading neighbors. This last feature is especially important in deep space applications, where global measurements are not present.

Second, we modify the dynamically-extended control laws to allow convergence to elliptical trajectories in three dimensions. Elliptical trajectories play a key role in space applications, since they are natural trajectories of the relative motion between vehicles in a rotating frame around the earth.

Finally, we discuss potential applications of our algorithms to the problem of distributed interferometric imaging with a satellite constellation, and we present the experimental results obtained on the 4DOF SPHERES testbed at MIT.

II. Mathematical Background

In this section, we provide some definitions and results from matrix theory which will be later applied to analyze the proposed control strategy.

A. Block Circulant Matrices

Let $A_1, A_2, \ldots, A_n$ be square matrices each of order m. A *block circulant* matrix of type (m, n) is a matrix of order mn of the form:

$$\hat{A} = \begin{pmatrix} A_1 & A_2 & \ldots & A_n \\ A_n & A_1 & \ldots & A_{n-1} \\ \vdots & \vdots & & \vdots \\ A_2 & A_3 & \ldots & A_1 \end{pmatrix}.$$

The entire matrix is determined by the first block row and we can write:

$$\hat{A} = \mathrm{circ}[A_1, A_2, \ldots, A_n].$$

B. Kronecker Product

Let A and B be $m\times n$ and $p\times q$ matrices, respectively. Then, the Kronecker product $A \otimes B$ of A and B is the $mp \times nq$ matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & \ldots & a_{1n}B \\ & & \\ \vdots & & \vdots \\ a_{m1}B & \ldots & a_{mn}B \end{pmatrix}.$$

It can be shown that if λ_A is an eigenvalue of A with associated eigenvector ν_A and λ_B is an eigenvector of B with associated eigenvector ν_B, then $\lambda_A\lambda_B$ is an eigenvalue of $A\otimes B$ with associated eigenvector $\nu_A\otimes\nu_B$.

Moreover, the following property holds: $(A\otimes B)(C\otimes D) = AC \otimes BD$, where A, B, C and D are matrices with the appropriate dimensions.

C. Determinant of matrix of submatrices

It can be shown that if A, B ,C and D are matrices of size $p \times n$, $p \times n$, $n \times p$ and $n \times p$ respectively and $AC = CA$ Then,

$$\det\left(\left[\begin{array}{cc} A & B \\ C & D \end{array}\right]\right) = \det(AD - CB) \qquad (1)$$

D. Rotation Matrices

A rotation matrix in $\mathbb{R}^3$ is a real square matrix whose transpose is its inverse and whose determinant is +1. Its eigenvalues are 1, $e^{i\theta}$ and $e^{-i\theta}$, where θ is the magnitude of the rotation about the rotation axis. For a rotation about the axis $(0, 0, 1)$, the corresponding eigenvectors are $(0, 0, 1)'$, $(1, +i, 0)'$, $(1, -i, 0)'$.

III. Cyclic Pursuit for Single Integrators

Consider in the three-dimensional space n mobile agents (uniquely labelled by an integer $i \in 1, 2, \ldots, n$), where agent i pursues the next $i + 1$, modulo n. Let $x_i(t) = [x_{i,1}(t), x_{i,2}(t)]^T \in \mathbb{R}^2$ be the position at time $t \geq 0$ of the i^{th} agent, where $i \in 1, 2, \ldots, n$. The kinematics of each agent is described by a simple integrator [15]:

$$\begin{aligned} \dot{x}_i &= u_i \\ u_i &= R(\alpha)(x_{i+1} - x_i), \end{aligned}$$

where $R(\alpha)$ is the rotation matrix:

$$R(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}.$$

The overall dynamics of the n agents is:

$$\dot{x} = k\hat{A}x, \qquad (2)$$

where $x = [x_1^T, x_2^T, \ldots, x_n^T]^T$ and $\hat{A}$ is the block circulant matrix (i.e.) $\hat{A}$ =circ$[-R(\alpha), R(\alpha), 0_{2\text{x}2}, \ldots, 0_{2\text{x}2}]$.

One can prove the following (see [15]).

Theorem 1: Consider the vector field (2). Then, agent positions starting at any initial condition in $\mathbb{R}^{2n}$ and evolving under (2) exponentially converge:

1) if $0 \leq \alpha < \pi/n$, to a single limit point: their initial center of mass;
2) if $\alpha = \pi/n$, to an evenly spaced circle formation;
3) if $\pi/n < \alpha < 2\pi/n$, to an evenly spaced logarithmic spiral formation.

Remark 2: The previous results can be easily generalized to three dimensions. Indeed, let $x_i(t) = [x_{i,1}(t), x_{i,2}(t), x_{i,3}(t)]^T \in \mathbb{R}^3$ be the position at time $t \geq 0$ of the i^{th} agent, where $i \in 1, 2, \dots, n$, and let $x = [x_1^T, x_2^T, \dots, x_n^T]^T$. Then, the dynamics of the overall system can be written as:

$$\dot{x} \;=\; (L \otimes R)x \tag{3}$$

where L is the circulant matrix:

$$L \;=\; \begin{bmatrix} \text{-1} & 1 & 0 & ... & 0 \\ 0 & \text{-1} & 1 & ... & 0 \\ . & . & . & . & \\ 1 & 0 & .. & 0 & \text{-1} \end{bmatrix},$$

and R is the rotation matrix. It can be shown (see [15]) that the eigenvalues of matrix L are $e^{2\pi kj/n} - 1$, where j is the imaginary unit and $k = 0, .., n-1$. Notice that one eigenvalue is zero.

By the properties of the Kronecker product, the $3n$ eigenvalues of $L \otimes R$ are:

$$\begin{aligned} \lambda_k &= e^{2\pi kj/n} - 1 && \text{for } k = 1, \dots, n \\ \lambda_k &= e^{2\pi(k-n)j/n + j\theta} - e^{j\theta} && \text{for } k = n+1, \dots, 2n \\ \lambda_k &= e^{2\pi(k-2n)j/n - j\theta} - e^{-j\theta} && \text{for } k = 2n+1, \dots, 3n \end{aligned} \tag{4}$$

Then, by identical arguments as in [15], it is possible to state, for the three-dimensional case, a Theorem equal to Theorem 1.

IV. Dynamic Cyclic Pursuit

In this section, we extend the previous cyclic pursuit control laws to double integrators; in other words, we consider a *dynamic* model for the agents. We first present a control law that requires each agent to be able to measure its velocity; then, we design a control law that only requires *relative* measurements of position and velocity. The trajectories are shown to converge trajectories in a plane perpendicular to the $(0, 0, 1)$ axis, this is not a limitation however, as it will be shown in a following section, the motion can be generalized to other planes of rotation.

In the following, let $x_i(t) = [x_{i,1}(t), x_{i,2}(t), x_{i,3}(t)]^T \in \mathbb{R}^3$ be the position at time $t \geq 0$ of the i^{th} agent, where $i \in 1, 2, \dots, n$. The dynamics of each agent is described by the double integrator:

$$\ddot{x}_i = u_i. \tag{5}$$

A. Dynamic Cyclic Pursuit with Reference Velocity

In [15], we proposed, without analyzing, the following control law for the double integrator dynamics (5):

$$u_i = k_d(R(x_{i+1} - x_i) - \dot{x}_i) + k_d R(\dot{x}_{i+1} - \dot{x}_i) \tag{6}$$

where k_d is a positive constant, and R is the rotation matrix about the $(0, 0, 1)$ axis.

The overall dynamics of the n agents is described by:

$$\frac{d}{dt}\begin{bmatrix} \text{p} \\ \text{v} \end{bmatrix} = \begin{bmatrix} 0 & I_{3n} \\ k_d(L \otimes R) & L \otimes R - k_d I_{3n} \end{bmatrix} x \doteq Cx \tag{7}$$

where $p = [x_1^T, x_2^T, \dots, x_n^T]^T$, $v = [\dot{x}_1^T, \dot{x}_2^T, \dots, \dot{x}_n^T]^T$, R is a rotation matrix, I_{3n} is the $3n$-by-$3n$ identity matrix, and L is the *circulant* matrix defined previously. The following theorem characterizes eigenvalues and eigenvectors of C.

Theorem 3: The eigenvalues of the state matrix C in Eq. (7) are the union of:

- the $3n$ eigenvalues of $L \otimes R$,
- -k_d (that has multiplicity $3n$).

That is $\lambda(C) = \lambda(L \otimes R) \cup \{-k_d\}$. Moreover, the eigenvector corresponding to the kth eigenvalue $\lambda_k \in \lambda(L \otimes R)$, $k = 1, \dots, 3n$, is:

$$\nu_k \doteq \begin{bmatrix} \nu_{p_k} \\ \nu_{v_k} \end{bmatrix} \;=\; \begin{bmatrix} \mu_k \\ \lambda_k \mu_k \end{bmatrix}, \quad k = 1, \dots, 3n,$$

where μ_k is the eigenvector of $L \otimes R$ corresponding to λ_k. The $3n$ (independent) eigenvectors corresponding to the eigenvalue $-k_d$ (that has multiplicity $3n$) are

$$\nu_k \doteq \begin{bmatrix} \nu_{p_k} \\ \nu_{v_k} \end{bmatrix} \;=\; \begin{bmatrix} k_d^{-1} e_{k-3n} \\ e_{k-3n} \end{bmatrix}, \; k = 3n+1, \dots, 6n,$$

where e_j is the jth vector of the canonical basis in $\mathbb{R}^{3n}$.

Proof: First, we compute the eigenvalues of C. The eigenvalues λ are, by definition, solutions of the characteristic equation:

$$0 \;=\; \det \begin{bmatrix} \lambda I_{3n} & -I_{3n} \\ -k_d(L \otimes R) & \lambda I_{3n} - (L \otimes R - k_d I_{3n}) \end{bmatrix}.$$

Using the result in Eq.(1) (clearly $I_{3n}(L \otimes R) = (L \otimes R)I_{3n}$), we have that

$$\begin{aligned} 0 &= \det\left(\lambda^2 I_{3n} - \lambda(L \otimes R - k_d I_{3n}) - k_d(L \otimes R)\right) \\ &= \det((\lambda I_{3n} + k_d I_{3n})(\lambda I_{3n} - (L \otimes R))) \\ &= \det((\lambda + k_d) I_{3n}) \det(\lambda I_{3n} - (L \otimes R)). \end{aligned}$$

Thus, the eigenvalues of C must satisfy:

$$\begin{aligned} 0 &= \det((\lambda + k_d) I_{3n}), \\ 0 &= \det(\lambda I_{3n} - (L \otimes R)); \end{aligned}$$

then, the first part of the claim is proven.

By definition, the eigenvector $[\nu_{p_k}\ \nu_{v_k}]^T$ corresponding to the eigenvalue λ_k, $k = 1, \dots, 6n$, satisfies the eigenvalue equation:

$$\lambda_k \begin{bmatrix} \nu_{p_k} \\ \nu_{v_k} \end{bmatrix} = \begin{bmatrix} 0 & I_{3n} \\ k_d(L\otimes R) & L\otimes R - k_d I_{3n} \end{bmatrix} \begin{bmatrix} \nu_{p_k} \\ \nu_{v_k} \end{bmatrix} = \begin{bmatrix} \nu_{v_k} \\ k_d(L\otimes R)\nu_{p_k} + (L\otimes R)\nu_{v_k} - k_d\nu_{v_k} \end{bmatrix}$$

Thus, we obtain

$$\begin{aligned} \lambda_k \nu_{p_k} &= \nu_{v_k}, \\ \lambda_k \nu_{v_k} &= k_d(L\otimes R)\nu_{p_k} + (L\otimes R)\nu_{v_k} - k_d\nu_{v_k}, \end{aligned}$$

and therefore

$$\lambda_k(k_d + \lambda_k)\nu_{p_k} = (k_d + \lambda_k)(L\otimes R)\nu_{p_k}.$$

If $\lambda_k = -k_d$, then we have $3n$ eigenvectors given by $[k_d^{-1}e_j,\ e_j]^T$, $j = 1, \dots, 3n$. If, instead, $\lambda_k \in \lambda(L\otimes R)$, we obtain from Eq. (8)

$$\lambda_k \nu_{p_k} = (L\otimes R)\nu_{p_k},$$

and we obtain the claim. ■

Assume, now, that R is about the rotation axis $(0, 0, 1)$; then it is possible to prove the following theorem.

Theorem 4: Consider the vector field (6). Then, agent positions starting at any initial condition in $\mathbb{R}^{3n}$ and evolving under (6) exponentially converge:

1) if $0 \le \theta < \pi/n$, to a single limit point: their initial center of mass;
2) if $\theta = \pi/n$, to an evenly spaced circle formation;
3) if $\pi/n < \theta < 2\pi/n$, to an evenly spaced logarithmic spiral formation.

Proof: The proof of this theorem reduces to verify that for $\theta < \pi/n$ there are three zero eigenvalues while all other eigenvalues are in the open left-half complex plane; for $\theta = \pi/n$ there are 3 zero eigenvalues, two eigenvalues are on the imaginary axis, while all other eigenvalues are in the open left-half complex plane; for $\pi/n < \theta < 2\pi/n$ there are 3 zero eigenvalues, two eigenvalues are on the open right-half complex plane, while all other eigenvalues are in the open left-half complex plane.

Indeed, the proof of the previous statements is a rather simple consequence of Theorem 3 and Theorem 3.2 in [15], and, thus, it is omitted in the interest of brevity. ■

B. Control Law with only Relative Information

Consider the following control law:

$$u_i = k_1 R^2\Big((x_{i+2} - x_{i+1}) - (x_{i+1} - x_i)\Big) + k_2 R(\dot{x}_{i+1} - \dot{x}_i) \quad (8)$$

where R is, as usual, a rotation matrix, and k_1, k_2 are two real constants (not necessarily positive). In this case, each agent needs to measure *relative* positions involving neighbors $i+1$ and $i+2$, and its *relative* velocity with respect to agent $i+1$. Notice that control law (8) does not require any knowledge of global position or velocity.

Notice that

$$L^2 = \begin{bmatrix} 1 & -2 & 1 & 0 & \dots & 0 \\ 0 & 1 & -2 & 1 & \dots & 0 \\ . & . & . & . & & \\ -2 & 1 & 0 & .. & 0 & 1 \end{bmatrix}$$

Then, the overall dynamics of the n agents is described by:

$$\dot{x} = \frac{d}{dt}\begin{bmatrix} \text{p} \\ \text{v} \end{bmatrix} = \begin{bmatrix} 0 & I_{3n} \\ k_1 L^2\otimes R^2 & k_2(L\otimes R) \end{bmatrix} x \doteq Fx \quad (9)$$

The following theorem characterizes eigenvalues and eigenvectors of F.

Theorem 5: The eigenvalues of the state matrix F in Eq. (9) are the union of:

- the $3n$ eigenvalues of $L\otimes R$, each one multiplied by β_+,
- the $3n$ eigenvalues of $L\otimes R$, each one multiplied by β_-,

where

$$\beta_\pm = \frac{k_2}{2} \pm \sqrt{\left(\frac{k_2}{2}\right)^2 + k_1}.$$

That is $\lambda(F) = \beta_+\lambda(L\otimes R) \cup \beta_-\lambda(L\otimes R)$, where $\beta_+\lambda(L\otimes R)$ means that each eigenvalue in the set $\lambda(L\otimes R)$ is multiplied by β_+ (similar considerations hold for $\beta_-\lambda(L\otimes R)$). Moreover, the eigenvector corresponding to the kth eigenvalue $\lambda_k \in \beta_+\lambda(L\otimes R)$, $k = 1, \dots, 3n$, is:

$$\nu_k \doteq \begin{bmatrix} \nu_{p_k} \\ \nu_{v_k} \end{bmatrix} = \begin{bmatrix} \mu_k \\ \lambda_k\mu_k \end{bmatrix}, \quad k = 1, \dots, 3n,$$

where μ_k is the eigenvector of $(L\otimes R)$ corresponding to λ_k/β_+. Similarly, the eigenvector corresponding to the kth eigenvalue $\lambda_{3n+k} \in \beta_-\lambda(L\otimes R)$, $k = 1, \dots, 3n$, is:

$$\nu_{3n+k} \doteq \begin{bmatrix} \nu_{p_{3n+k}} \\ \nu_{v_{3n+k}} \end{bmatrix} = \begin{bmatrix} \mu_k \\ \lambda_k\mu_k \end{bmatrix}, \quad k = 1, \dots, 3n,$$

where μ_k is the eigenvector of $(L\otimes R)$ corresponding to λ_k/β_-.

Proof: First, we compute the eigenvalues of F. Notice that $L^2\otimes R^2 = (L\otimes R)^2$. The eigenvalues λ are, by definition, solutions of the characteristic equation:

$$0 = \det\left(\begin{bmatrix} \lambda I_{3n} & -I_{3n} \\ -k_1(L\otimes R)^2 & \lambda I_{3n} - k_2(L\otimes R) \end{bmatrix}\right)$$

Using the result in Eq.(1) (clearly $I_{3n}(L \otimes R)^2 = (L \otimes R)^2 I_{3n}$), we have that

$$
\begin{aligned}
0 &= \det\left(\lambda^2 I_{3n} - k_2\lambda(L \otimes R) - k_1(L \otimes R)^2\right) \\
&= \det\Big((\lambda I_{3n} - \beta_+(L \otimes R))(\lambda I_{3n} - \beta_-(L \otimes R))\Big) \\
&= \det(\lambda I_{3n} - \beta_+(L \otimes R)) \det(\lambda I_{3n} - \beta_-(L \otimes R))
\end{aligned}
$$

Thus, the eigenvalues of F must satisfy

$$
\begin{aligned}
0 &= \det(\lambda I_{3n} - \beta_+(L \otimes R)), \\
0 &= \det(\lambda I_{3n} - \beta_-(L \otimes R)).
\end{aligned}
$$

Then, the first part of the claim is proven.

By definition, the eigenvector $[\nu_{p_k}\ \nu_{v_k}]^T$ corresponding to the eigenvalue λ_k, $k = 1, \ldots, 6n$, satisfies the eigenvalue equation:

$$
\begin{aligned}
\lambda_k \begin{bmatrix} \nu_{p_k} \\ \nu_{v_k} \end{bmatrix} &= \begin{bmatrix} 0 & I_{3n} \\ k_1(L \otimes R)^2 & k_2(L \otimes R) \end{bmatrix} \begin{bmatrix} \nu_{p_k} \\ \nu_{v_k} \end{bmatrix} \\
&= \begin{bmatrix} \nu_{v_k} \\ k_1(L \otimes R)^2 \nu_{p_k} + k_2(L \otimes R)\nu_{v_k} \end{bmatrix}
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
\lambda_k \nu_{p_k} &= \nu_{v_k}, \\
\lambda_k \nu_{v_k} &= k_1(L \otimes R)^2 \nu_{p_k} + k_2(L \otimes R)\nu_{v_k},
\end{aligned}
$$

and therefore,

$$
\lambda_k^2 \nu_{p_k} = k_1(L \otimes R)^2 \nu_{p_k} + k_2(L \otimes R)\lambda_k \nu_{p_k}
$$

which can be rewritten as

$$
(\lambda_k I_{3n} - \beta_+(L \otimes R))(\lambda_k I_{3n} - \beta_-(L \otimes R))\nu_{p_k} = 0. \quad (10)
$$

Therefore, if $\lambda_k \in \beta_+\lambda(L \otimes R)$, the above equation is satisfied by letting ν_{p_k} be equal to μ_k, since (notice that μ_k is the eigenvector of $(L \otimes R)$ corresponding to eigenvalue λ_k/β_+):

$$
\lambda_k \nu_{p_k} = \frac{\lambda_k}{\beta_+}\beta_+\mu_k = \beta_+(L \otimes R)\mu_k = \beta_+(L \otimes R)\nu_k
$$

By identical arguments, if $\lambda_k \in \beta_-\lambda(L \otimes R)$, the Eq. (10) is satisfied by letting ν_{p_k} be equal to μ_k. Thus, the theorem is proven. ■

By appropriately choosing k_1, k_2 and R, it is possible to obtain a variety of formations. The following Theorem shows how to achieve circular formations.

Theorem 6: Let $k_2 = 2$ and $k_1 = -\left(\frac{k_2}{2}\right)^2 - \tan^2 \frac{\pi}{2n}$. Moreover, assume that R a $\pi/(2n)$-rotation matrix with rotation axis $(0, 0, 1)$; then, the system converges to a circular formation.

Proof: Then, from Eq. (4) and Theorem 9, we argue that two eigenvalues are on the imaginary axis, six eigenvalues are zero, and all other eigenvalues are in the open left half complex plane. As a results, the system will converge to a circular formation. ■

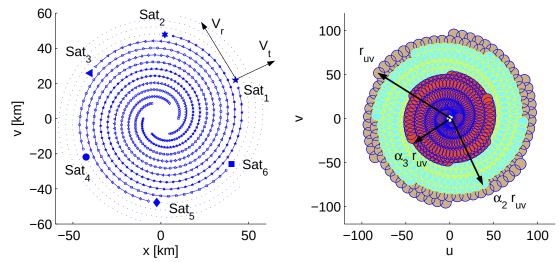

Fig. 1. u-v plane coverage by a system of multiple spacecraft.

V. Applications Of Cyclic Pursuit: Coverage of the u-v plane

So far, the cyclic pursuit strategy has received considerable attention in the control community, despite the fact that, to the best of our knowledge, no application has been presented for which cyclic pursuit is the best solution. In this section, we propose the first application in which cyclic pursuit is an ideal candidate control law.

Interferometric imaging, the image reconstruction from interferometric pattern, is an application of formation flight that has been devised and studied for missions such as NASA's TPF [5] and ESA's Darwin [6]. The general problem of interferometric imaging consists of performing measurements in a region of the observation plane in such a way that enough information about the spatial frequency content of the image is obtained. A heuristic solution to this coverage problem has been proposed to be the set of *archimedes spiral* trajectories [11], where the baselines for each pair of sensors describe "coverage discs" as shown in Fig.1. The coverage requirement is a nonlinear function that conveys the fact that the trajectories of the "coverage discs" should allow for a minimum ammount of time to be spent at each region of the u-v plane to be covered. This coverage constraint has been written as [11]:

$$
\sum_{i}^{N_R} \frac{b a_i}{V_r(a_i p)} \geq \bar{C} \quad (11)
$$

Where N_R is the number of rings formed by an specific configuration, and $\bar{C}$ is a parameter of the image that defines the total amount of light collection time for each spatial frequency. b has the value 1 for odd total number of vehicles and 2 for even.

The application of the described cyclic pursuit algorithm to uv plane coverage is inherently appropriate given the fact that the frequency plane coverage problem is independent of the global location of the spacecraft, it lies only on the space of relative positions, and thus,

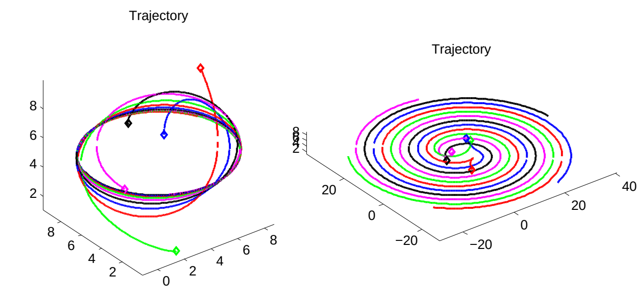

Fig. 2. Convergence from random initial conditions to symmetric formations. Left:Circular trajectories , Right:Archimedes spiral: the gains selected such that, the separation between consecutive coverage discs is appropriate and the tangential velocity of the vehicles meet the coverage requirement

the fact that the dynamics show unstable modes (eigenvalues=0) does not play importance for the achievement of the task. Additionally, missions like TPF and Darwin, consider locations far out of the reach of GPS signals and are expected to rely on relative measurements to perform reconfigurations and observation maneuvers.

Figures 2 show the resulting trajectories for two cases where the vehicles are initialy located at random inside a volume of $(10km)^3$. In one case converging to circular trajectories, in other describing archimedes spirals. The inertial frame for the plots is the geometric center of the configuration.

VI. On Reaching Natural Trajectories

In this section we modify the previous control laws to allow convergence to elliptical trajectories. Consider the application of a similarity transformation to the rotation matrix. For simplicity, consider for the single integrator model the control law:

$$u_i \;=\; TRT^{-1}(x_{i+1} - x_i) \tag{12}$$

where T is a linear transformation, e.g:

$$T \;=\; \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ z_0\cos(\phi_z) & z_0\sin(\phi_z) & 1 \end{bmatrix}. \tag{13}$$

A similarity transformation does not change the eigenvalues of the system, so the previous stability analysis holds. The eigenvectors are transformed according to $\nu' = T\nu$. In particular, if R is a π/n rotation matrix with rotation axis $(0,0,1)^T$, then the system, instead of converging to a circular formation, will converge to ellipses with an $x : y$ ratio of $1 : 2$ and $y : z$ ratio of $1 : z_0$, and the phase between the motion in x and the motion in z will be ϕ_z.

Indeed, the above approach is useful to allow the system to globally (i.e., from any initial condition) converge to *natural* trajectories. Consider the dynamic system:

$$\begin{aligned} \ddot{x} &= f(x,\dot{x}) - u \\ &= f(x,\dot{x}) - \hat{f}(x,\dot{x}) + u_k \end{aligned}$$

which has a zero-effort (u=0) invariant set $x = x^*$, such that $\hat{f}(x^*, \dot{x}^*) = u_k$. If we have a controller such that the state reach $x^*(t)$ as $t \to \infty$, then the control effort will tend to $u = 0$ as $t \to \infty$. In the case of dynamics in relative orbits, elliptical relative trajectories are closed near-natural trajectories (i.e. can be maintained with little or no control effort).

A. *Clohessy-Wiltshire model*

The Clohessy-Wiltshire model approximates the motion of a satellite with respect to a rotating frame that follows a circular orbit with angular velocity n and radius $R_{ref} = (\mu/n^2)^{1/3}$. The equations of motion are:

$$\begin{aligned} \ddot{x} &= 2n\dot{y} + 3n^2x + u_x \\ \ddot{y} &= -2n\dot{x} + u_y \\ \ddot{z} &= -n^2z + u_z \end{aligned} \tag{14}$$

This set of equations have natural solutions (u=0):

$$\begin{aligned} x^*(t) &= \frac{1}{2}x_o\sin(nt+\psi) \\ y^*(t) &= x_o\cos(nt+\psi) \\ z^*(t) &= z_o\sin(nt+\psi+\phi_z) \end{aligned} \tag{15}$$

which is the set of ellipses with ratio x:y = 1:2 and angular velocity n and can be expressed as:

$$x^* \;=\; T\begin{bmatrix} x_o\sin(nt) \\ x_o\cos(nt) \\ z_o\sin(nt) \end{bmatrix} = T\xi^*(t) \tag{16}$$

A cyclic pursuit controller:

$$\begin{aligned} u_i \;=\; & -\hat{f}_{cw}(x_i,\dot{x}_i) + k_d(C(x_{i+1}-x_i) - \dot{x}_i) \\ & + k_dC(\dot{x}_{i+1} - \dot{x}_i) \end{aligned} \tag{17}$$

with $C = K_rTRT^{-}1$, T defined as in Eq. (13), and $K_r = n/(2\sin(\pi/N))$, will make the vehicles converge to an elliptical formation, with ratio x:y = 1:2 and

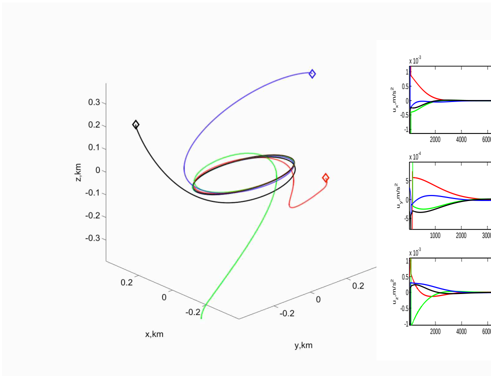

Fig. 3. Convergence to elliptical trajectories in CW dynamics. Right: control effort: as $t \to \infty$, $u \to 0$

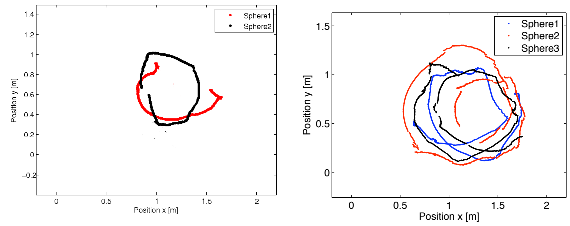

Fig. 4. Experiments 2 and 3 vehicles.

angular rate:

$$\begin{aligned}\omega &= \frac{|\dot{\xi}_i|}{|\xi_i|} = \frac{|T^{-1}\dot{x}|}{|T^{-1}x_i|} \\ &= \frac{n|T^{-1}TRT^{-1}(x_{i+1}-x_i)|}{2\sin(\pi/N)|\xi_i|} \\ &= \frac{n|R(\xi_{i+1}-\xi_i)|}{2\sin(\pi/N)|\xi_i|} \\ &= \frac{n|\xi_{i+1}-\xi_i|}{2\sin(\pi/N)|d|} = \frac{n|d|}{|d|} = n\end{aligned}$$

which means that the elliptical trajectories to which the vehicles converge in the physical space $x^*(t) = T\xi^*(t)$ are those in eq.(15).

B. Results

Figure 3 shows simulation results for the control laws described in this section. The system is simulated using the Clohessy-Wiltshire dynamics described in section IV.A. The results show convergence from random initial positions to the desired orbits, with equal phasing, as well as in the desired planes. It is also shown how the control effort goes to zero as the vehicles reach the desired *natural* trajectories.

VII. Hardware Implementation

A experimental demonstration of the algorithm was performed using the SPHERES testbed. SPHERES is an experimental testbed consisting of a group of small vehicles with the basic functions of a satellite. Three of them are on board of the International Space Station and three of them are ground based and perform 2D maneuvers sliding on air pucks on a flat table. The propulsion systems is compressed CO_2 gas. The metrology system calculates a global estimate by using ultrasound pulses. Each vehicle has a local estimator that calculates its global position. A single RF channel is used to communicate the state to a neighbor spacecraft, this communication is TDMA based, and allows for a small bandwidth of transmission in each period of control. The controller was tested on configurations of two and three vehicles on the ground testbed, where realistic effects like time delays in control and communication, noisy measurements, estimation filter lags, saturation of thrusters and large external disturbances (mainly friction with the table and gravity) affected the performance. The control period and estimation period in each vehicle is $\tau = 1$ second. The experimental results demonstrate the suitability of the proposed cyclic pursuit controller to achieve revolving trajectories. Moreover, these results represent a first implementation of cyclic pursuit in a holonomic system which approximates the dynamics of spacecraft in deep space. Videos of experimental results can be accessed at *http://ssl.mit.edu/spheres/video/UVcoverage/*.

VIII. Nonlinear extensions: Maintaining a constant intervehicle distance

A nonlinear extension of the previous control law can be set up by defining a state dependent rotation matrix. We can define the angle θ of the rotation matrix as

$$\theta = \pi/N + K_\alpha((\bar{p}_d - |\chi_{i+1} - \chi_i|));$$

such choice gives rise to expanding spirals if the intersatellite distance is less than the one desired, contracting spirals if the distance is larger than the one required, and circular trajectories if the intersatellite distance is the one required. Furthermore, an adaptive term can be added to the control law:

$$\begin{aligned}\theta &= K_{ir}\int((\bar{p}_d - |\chi_{i+1} - \chi_i|))dt \\ &\quad + K_\alpha((\bar{p}_d - |\chi_{i+1} - \chi_i|))\end{aligned}$$

such term alloes converge to circular formations even in the absence of knowledge of the number of vehicles in the formation.

The proof of this nonlinear law is out of the scope of the paper and has proven to be quite elusive, however, extensive simulation shows a very large region of convergence.

IX. Conclusions

In this paper we have proposed novel control laws and applications within the framework of cyclic pursuit. Possible future research directions include the study of convergence to more complex trajectories as Archimedean spirals, analysis of the proposed nonlinear control laws, and experiments on board of the International Space Station.

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