Decentralized Vehicle Routing in a Stochastic and Dynamic Environment with Customer Impatience

M. Pavone, N. Bisnik, E. Frazzoli, V. Isler

***Abstract*— Consider the following scenario: a spatio-temporal stochastic process generates service requests, localized at points in a bounded region on the plane; these service requests are fulfilled when one of a team of mobile agents visits the location of the request. For example, a service request may represent the detection of an event in a sensor network application, which needs to be investigated on site. Once a service request has been generated, it remains active for an amount of time which is itself a random variable, and then expires. The problem we investigate is the following: what is the minimum number of mobile agents needed to ensure that each service request is fulfilled before expiring, with probability at least $1-\varepsilon$? What strategy should they use to ensure this objective is attained? Formulating the probability of successfully servicing requests before expiration as a performance metric, we derive bounds on the minimum number of agents required to ensure a given performance level, and present decentralized motion coordination algorithms that approximate the optimal strategy.**

***Index Terms*— Mobile Robotic Networks, Sensor Networks, Traveling Salesman Problem.**

I. INTRODUCTION

Imagine the following scenario: a sensor network composed of a large number of nodes is deployed over a vast field, for example to study the behaviors of elusive animals, or to detect suspicious activity in a protected region as, for example, home burglaries, or insurgents placing Improvised Explosive Devices (IEDs). Typically, network nodes contain inexpensive sensors, such as motion detectors, which are susceptible to false alarms. Suppose that, in addition to the sensor network, a team of Unmanned Aerial Vehicles (UAVs) is also available, which are equipped with more sophisticated on-board sensors. Each time a sensor detects an event, a UAV is sent to the location to investigate the cause of the alarm, i.e., to track the animal or the intruders. Timeliness in such applications is of primary importance: should the UAV take too long to reach the location of the event, its cause may have already left the premises, and be hard to track.

This scenario can be formulated as the following problem: a group of vehicles is charged with servicing stochastically generated demands in an environment. Each demand appears at a random location (sampled from a known distribution) and is serviced when a vehicle moves to that location. In this paper, we study a novel version of this problem where the demands expire after a certain amount of time. We assume that the lifetime of a demand is randomly chosen from a known distribution and seek the minimum number of vehicles (and their control strategies) to maximize the probability that a demand is serviced before expiration, or equivalently, to maximize the percentage of successfully serviced demands.

M. Pavone and E. Frazzoli are with the Laboratory for Information and Decision Systems, and the Aeronautics and Astronautics Department, of the Massachusetts Institute of Technology, Cambridge, MA 02139, USA; email: {pavone,frazzoli}@mit.edu.

N. Bisnik and V. Isler are with the Department of Computer Science of the Rensselaer Polytechnic Institute, Troy, NY 12180, USA; email: {bisnin, isler}@cs.rpi.edu.

A. Related work

Considerable research effort has been invested in studying coverage properties of static sensor networks [1]–[5].

More recently, there has been growing interest in understanding how the coverage properties of a sensor network may be improved by introducing mobility to the sensor devices. The problem of relocating sensors to improve coverage has been studied in [6]. In this formulation, the sensors can individually estimate the positions of the targets. However, the quality of coverage decreases with increasing distance. In [7] and [8], the authors propose virtual force based algorithms in order to guide sensor movements for improving the coverage properties after random deployment. In [9], the authors propose algorithms to detect the vacancies in a sensor field and use them to guide sensor motion in order to increase coverage. The average area covered by mobile sensors over a period of time has been characterized in [10]. It is shown that for a mobile sensor network with density λ, with each sensor moving according to a mobility model similar to random walk with expected velocity $E[V_s]$, the expected area covered in time interval $(0,t)$ is given by $1-\exp\left(-\lambda(\pi r^2+2rE[V_s]t)\right)$.

A closely related dynamic vehicle routing problem is considered in [11]–[13]. In the dynamic vehicle routing problem, mobile agents are required to visit target points generated dynamically by a stochastic process. In [13], it is shown that the lack of communication between the mobile agents does not effect the performance of the system; however, the lack of communication slows the rate of convergence to the steady state. Bisnik *et al.* studied a similar vehicle routing problem where demands disappear [14]. They presented approximation algorithms for the case where the vehicles motion is restricted to a planar curve.

The problem of site visitation using multiple robots has also been studied in context of sweeping, covering, or exploration planar regions [15]–[18]. An offline algorithm for sweeping a known area by multiple mobile robots is proposed in [17].

B. Statement of Contributions

We study the version of this problem where the event location is chosen uniformly at random in the environment and the event lifetimes are independent and identically distributed according to an arbitrary distribution, thus extending our complementary previous results, e.g., from [12] and [14]. We present a constant factor approximation algorithm to compute the minimum number of vehicles required to guarantee that each event is serviced with probability at least $1-\varepsilon$ where ε is an "accuracy" parameter input to the problem. We also present a distributed strategy for assigning events to the vehicles, and to route them in an efficient way. Aside from applications in monitoring and surveillance, we believe that our results and techniques will be of independent interest due to their relation to the fundamental Traveling Salesperson Problem (TSP), and its stochastic and dynamic versions, such as the Dynamic Traveling Repairperson Problem (DTRP).

C. Paper organization

The paper is structured as follows. In Section II we introduce some background on convergence of random variables, the Euclidean Traveling Salesperson Problem, and Voronoi Diagrams. In Section III we present the problem formulation. In Section IV and V we compute a lower bound on the minimum number of agents needed to meet the objective as stated in Section III, and develop *centralized* routing algorithms providing a constant-factor approximation to the optimal strategy. In Section VI we present results from numerical experiments, and in Section VII we modify the previous centralized routing algorithms to make them *decentralized*. Finally, in Section VIII we draw some conclusions and discuss some directions for future work.

II. Preliminaries

In this section, we briefly describe some known concepts from probability and locational optimization, on which we will rely extensively later in the paper.

A. Almost Sure Convergence

A sequence of random variables $\{X_n\}$ converges almost surely to X ($\lim_{n\to\infty} X_n \overset{a.s.}{=} X$) if $\lim_{n\to\infty} X_n(\omega) = X(\omega)$ for all sample functions $\omega \in \Omega$ where $\mathbb{P}[\Omega] = 1$. (In other words, $\mathbb{P}[\lim_{n\to\infty} X_n = X]$=1.) The sequence of random variables $\{X_n\}$ converges almost surely to X if and only if, for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left[\bigcup_{k=n}^{\infty}[|X_k - X| > \varepsilon]\right] = 0.$$

B. Asymptotic Properties of the Traveling Salesperson Problem in the Euclidean Plane

The Traveling Salesperson Problem is one of the most widely known combinatorial and geometric optimization problems. In this section, we briefly review its Euclidean version and some results that are relevant to our analysis.

The Euclidean Traveling Salesperson Problem (TSP) is formulated as follows: given a set D of points in $\mathbb{R}^d$, find the minimum-length closed path (tour) through all points in D. Let $\mathrm{TSP}(D)$ denote the minimum length of a tour through all the points in D; by convention, $\mathrm{TSP}(\emptyset) = 0$.

The stochastic version of the Euclidean TSP enjoys some interesting properties. Let D_n be a set of n independent, identically distributed random variables, representing points in $\mathbb{R}^d$. Let each random variable in D_n be sampled from a compact set $Q \subset \mathbb{R}^d$, according to a distribution f. In [19] it is shown that there exists a constant $\beta_{\mathrm{TSP},d}$ such that

$$\lim_{n\to+\infty} \frac{\mathrm{TSP}(D_n)}{n^{1-1/d}} = \beta_{\mathrm{TSP},d} \int_Q \bar{f}(q)^{1-1/d}\, dq \quad \text{a.s.}, \qquad (1)$$

where $\bar{f}$ is the density of the absolutely continuous part of the distribution f. In other words, the optimal cost of stochastic TSP tours approaches a deterministic limit, and grows as the square root of the number of points in D; the current best estimate of the constant in the case $d = 2$ is $\beta_{\mathrm{TSP},2} \simeq 0.7120$ [20].

Notice that the bound (1) holds for all compact sets: the shape of the set only affects the convergence rate to the limit. According to [21], if Q is a "fairly compact and fairly convex" set in the plane, then Eq. (1) provides an adequate estimate of the optimal TSP tour length for values of n as low as 15. Remarkably, the asymptotic cost of the stochastic TSP for uniform point distributions is an upper bound on the asymptotic cost for general point distributions; this follows directly from an application of Jensen's inequality for concave functions to the right hand side of (1).

C. Voronoi Diagrams

An overview of Voronoi diagrams is presented in [22], [23], concepts and applications are discussed in [24] and abstract Voronoi diagrams are discussed in [25]. Let $\{g_1, \cdots, g_m\}$ be a collection of points in a convex compact set $\mathcal{Q}$ in a finite dimensional Euclidean space $\mathbb{R}^d$ (it is anyway possible to generalize the concept of Voronoi diagrams to any metric space), and let $\|\cdot\|$ denote the Euclidean norm in $\mathbb{R}^d$. Let the Voronoi region $V_i = V(g_i)$ be the set of all points $q \in Q$ such that $\|q - g_i\| \leqslant \|q - g_j\|$ for all $i \neq j$. The boundary of each V_i is a *convex polygon*. The set of regions $\{V_1, \cdots, V_m\}$ is called the Voronoi diagram for the generators $\{g_1, \cdots, g_m\}$. When the two Voronoi regions V_i and V_j are adjacent, g_i is called a *Voronoi neighbor* of g_j (and vice-versa). We also define the $(i,\, j)$-face as $\Delta_{ij} = V_i \bigcap V_j$. We will shortly refer to the vertices of a face Δ_{ij} as the set $\{u_l\}$, without any additional subscript: the hidden subscripts will be clear from the context. Voronoi diagrams enjoy the Perpendicular Bisector Property: the face Δ_{ij} bisects the line segment joining g_j and g_i and that line segment is perpendicular to the face. With reference to Fig. 1, we introduce the following notation (for Voronoi regions in $\mathbb{R}^2$): $O_{ij}^g = (g_j + g_i)/2$; $O_{ij}^u = (u_1 + u_2)/2$; $\gamma_{ij} = \|g_j - g_i\|$; $\delta_{ij} = \|u_2 - u_1\|$.

Finally, we define an equitable Voronoi diagram as a Voronoi diagram where all Voronoi cells have same measure.

III. Notation and problem formulation

In this section, we first describe the problem set-up. Next, we formulate the main problem studied in this paper.

A. Problem Set-Up

Let the environment $Q \subseteq \mathbb{R}^d$ be a convex, bounded set; for simplicity, we will mainly consider the planar case, i.e., $d = 2$, with the understanding that extensions to higher dimensions

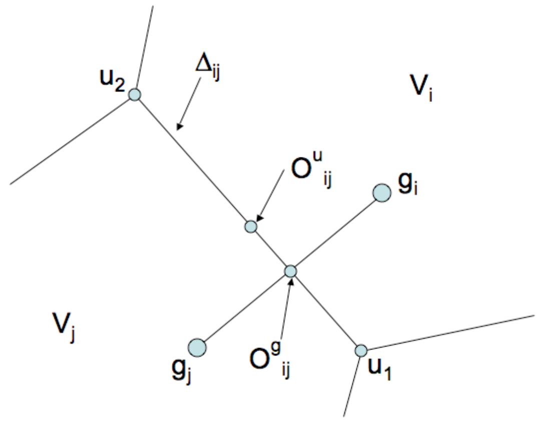

Fig. 1

NOTATION. THE POINTS g_i AND g_j ARE THE GENERATORS OF THE VORONOI REGIONS, RESPECTIVELY, V_i AND V_j.

are possible. Without loss of generality we will assume that the measure of Q (denoted as $|Q|$) is 1.

Demands are generated according to a homogeneous spatio-temporal Poisson Point process, with time intensity $\lambda > 0$, and spatial density $f : Q \to \mathbb{R}_+$. In other words, the number of demands generated over time within a region $S \subseteq Q$ can be described as a homogeneous Poisson process with rate

$$\lambda_S = \lambda \int_S f(q)\, dq.$$

Without loss of generality, we assume f to be normalized such that $\int_Q f(q)\, dq = 1$; in such case, f can be interpreted as a probability density function. In this paper, in particular, we will consider a uniform distribution for the demand locations, i.e., $f(q) = 1/|Q|$.

Let $D(t)$ be the set of locations of demands generated up to time t. Given a set $S \subseteq Q$, the expected number of demands generated in S within the time interval $[t, t+\Delta t]$ is

$$\mathbb{E}\left[\mathrm{card}(D(t+\Delta t) \cap S) - \mathrm{card}(D(t) \cap S)\right] = F(S)\lambda\Delta t,$$

where $F(S) \triangleq \int_S f(q)\, dq$. We will label demands in increasing order with respect to time of arrival; for the orderliness property of Poisson processes, this is a well-defined criterion.

Demands are serviced by a team of m holonomic vehicles, modeled as point masses. The vehicles are free to move, with bounded velocity, within the environment Q; without loss of generality, we will assume that the velocity magnitude is unitary. The vehicles are identical, and have unlimited fuel and demand servicing capacity. For simplicity, vehicles are not required to stop or to loiter in proximity of demands: extension to the case with additional on-site servicing time is straightforward, but the notation is more cumbersome. Thus, a demand is serviced as a vehicle visits its location.

Let $L_r > 0$, $r \in \mathbb{N}$ be a random variable describing the *impatience* of the r-th demand: in other words, should the r-th demand not be visited within time L_r from its arrival, it will expire. We assume that the impatience times L_r are independent and identically distributed according to a common density $f_L : \mathbb{R}_+ \to \mathbb{R}_+$.

Let W_r, $r \in \mathbb{N}$ be a random variable expressing the sojourn time in the system for the r-th demand. The random variable W_r is the elapsed time between the arrival of demand r and the time when either one of the servers completes its service or such demand departs from the system due to impatience. A demand is considered serviced if $W_r < L_r$.

Information on outstanding demands (i.e., arrived demands that have neither been serviced nor impatiently departed) at time t is summarized as a finite set of demand positions $D_\mathrm{o}(t) \subseteq D(t)$. In other words, demands are inserted in both D and D_o as soon as they are generated; they are removed from D_o either upon servicing—as a vehicle visits the demand's location— or upon expiration. We assume that information contained in $D_\mathrm{o}(t)$ is available to all vehicles.

B. Problem Formulation

Informally, the objective is to ensure that no more than a fraction ε (where $\varepsilon \in (0,1]$ is a control parameter) out of all the arrived demands departs impatiently before service. We will refer to ε as the "accuracy" of the system. In particular, we want to answer the questions: what is the minimum number of mobile agents needed to ensure that each service request is fulfilled before expiring, with probability at least $1-\varepsilon$? What strategy should they use to ensure this objective is attained?

To state our problem formulation rigorously, we first define the critical time T_crit as

$$T_\text{crit} = \max\Big\{T \in \mathbb{R}_+ : \int_T^\infty f_L(t)dt = \mathbb{P}\left[L > T\right] \geqslant 1-\varepsilon\Big\}.$$

Clearly, if a routing policy is able to ensure that *each* demand location (regardless of its impatience) is visited within time T_crit from its arrival, then this policy ensures that no more than a fraction ε out of all the arrived demands departs impatiently before service. Through the concept of critical time we can, therefore, address the problem of servicing demands *with* impatience as the problem of visiting *all* demands' locations, regardless of their impatience (i.e., even if they depart impatiently), within a constant time; this approach on the one hand introduces some degree of conservatism, but on the other hand it simplifies considerably the mathematical analysis.

With the above discussion in mind, define Γ as the set of all possible policies to visit all demands' locations regardless of their impatience. For such policies, define $\tilde{W}_r$ as a random variable expressing the elapsed time between the arrival of demand r and the time its location is visited; we call these random variables virtual waiting times. The problem becomes finding policies belonging to Γ that, with the minimum possible number m of agents, ensure: $\lim_{r\to\infty} \tilde{W}_r \overset{a.s.}{<} T_\text{crit}$. For technical reasons, we restrict our analysis to policies that satisfy the additional requirement that $\lim_{r\to\infty} \mathbb{E}\left[\tilde{W}_r\right] < T_\text{crit}$; this form of commutativity between limit and expectation will be crucial in Section IV.

Our problem formulation can, therefore, be stated as follows:

Definition 3.1 (Problem formulation): Find routing policies belonging to Γ that, with the minimum possible number m of agents, ensure:

$$\begin{aligned} &\text{(i)} \quad \lim_{r\to\infty} \tilde{W}_r \overset{a.s.}{<} T_{\text{crit}}; \\ &\text{(ii)} \quad \lim_{r\to\infty} \mathbb{E}\left[\tilde{W}_r\right] < T_{\text{crit}}. \end{aligned} \tag{2}$$

Let Π be the set of all control policies belonging to Γ able to guarantee Eq. (2). We proceed as follows: we first find in Section IV a lower bound on the number of vehicles needed by each policy belonging to the set Π. Then, in Section V, we analyze a policy that provides a constant factor approximation to the optimal policy.

We mention that a somehow less conservative approach would be to study the limit $\lim_{r\to\infty} \mathbb{P}[L_r = W_r]$ and require this limit to be less than ε; this approach is currently subject of ongoing research by the authors.

IV. Lower Bound

When we consider policies belonging to Γ, i.e., policies that visit all demands' locations regardless of their impatience, our problem is identical to the classical m-vehicle Dynamic Traveling Repairperson Problem (m-DTRP), with the exception that in our paper we want to enforce an upper bound, equal to T_{crit}, to the virtual waiting times. The m-DTRP problem has been extensively studied in [26]–[28]; thus, to find a lower bound on the number of agents we can use some of the known results about the m-DTRP problem. In particular, we make use of the following theorem [28]:

Theorem 4.1: Let be $\overline{W} = \lim_{r\to\infty} \mathbb{E}\left[\tilde{W}_r\right]$, where $\tilde{W}_r$ is the waiting time of demand r in the m-DTRP problem (identical, for policies belonging to Γ, to the virtual waiting times in our problem), and assume zero on-site service. Then there exists a constant γ such that:

$$\overline{W} \geqslant \gamma^2 \frac{\lambda}{m^2},$$

where $\gamma \geqslant 2/(3\sqrt{2\pi})$. In the limit $\lambda \to \infty$, the constant γ satisfies $\gamma \geqslant 2/(3\sqrt{\pi})$.

Considering that for a policy π belonging to Π we require $\overline{W} < T_{\text{crit}}$ (here the technical condition (ii) in Def. 2 becomes crucial), we get the following lower bound:

Theorem 4.2: A lower bound on the minimum number of vehicles needed by any policy π belonging to Π is:

$$m > \sqrt{\gamma^2 \frac{\lambda}{T_{\text{crit}}}}.$$

V. Constant factor approximation policy

In this section, we propose a policy that provides a constant factor approximation to the optimal policy in heavy load (i.e., for large λ). The light load analysis is much simpler and is not included in the paper.

Multiple-Vehicle TSP policy: at start-up, the environment Q is partitioned into m service regions Q_j of identical area $1/m$ (recall that $|Q| = 1$) and each agent is assigned to a distinct service region. Then, each agent executes in its own service region:

1) if there are no unvisited demands, move at unit speed toward the median of the service region;
2) if there are unvisited demands, do the following: (i) compute the TSP tour through all demands, (ii) service all the demands in such tour. No shortcuts are allowed if a demand departs impatiently before service. Repeat from point 1).

We will refer to the i-th time instant in which an agent computes a new TSP tour as the epoch i of the policy. From the definition of the policy, epochs evolve independently in each service region. We analyze this policy in the heavy load limit, i.e., in the limit $\lambda \to \infty$. For simplicity, we consider only countably infinite values of λ, i.e., we write $\lambda = kl$ with $k \in \mathbb{N}^+$ and $l > 0$ an arbitrary constant. Therefore, the heavy load limit is obtained for $k \to \infty$.

Let us study the waiting time of a demand within one of the m service regions Q_j, $j = 1, \cdots, m$. The arrival rate in each of these service regions, whose area is $1/m$, is $\bar{\lambda} = kl/m$. We introduce the following notation:

- $n_k^j(i)$: number of unvisited demands in service region j at epoch i (i.e., number of demands arrived between epoch $i-1$ and epoch i, $i > 1$, in service region j) when the overall arrival process has intensity $\lambda = kl$;
- $C^j(i)$: length of the TSP tour through the demands that are unvisited at epoch i in service region j. Since we are assuming a unitary velocity magnitude, $C^j(i)$ is also the time length of the time interval between epochs i and $i+1$ in service region j (since we consider heavy load conditions, we can safely neglect the travel component between the agent's current position and the closest demand in the TSP tour).

In the next lemma we show some preliminary limit results.

Lemma 5.1: The following limits hold with probability one in each service region j, at each epoch $i > 1$ and for any $m \in \mathbb{N}^+$:

$$\begin{aligned} &\text{(i)} && \lim_{k\to\infty} && n_k^j(i) && \overset{a.s}{=} && \infty; \\ &\text{(ii)} && \lim_{k\to\infty} && \frac{C^j(i)}{\sqrt{n_k^j(i)}} && \overset{a.s}{=} && \frac{\beta_{\text{TSP},2}}{\sqrt{m}}; \\ &\text{(iii)} && \lim_{k\to\infty} && C^j(i) && \overset{a.s}{=} && \infty; \\ &\text{(iv)} && \lim_{k\to\infty} && \frac{n_k^j(i+1)}{C^j(i)\frac{kl}{m}} && \overset{a.s}{=} && 1. \end{aligned}$$

Proof: See Appendix. ∎

We obtain the following asymptotic result:

Theorem 5.2: For any $m \in \mathbb{N}^+$ we have

$$\lim_{r\to\infty} \lim_{k\to\infty} \frac{\tilde{W}_r}{kl} \overset{a.s.}{\leqslant} 2\frac{\beta^2_{\text{TSP},2}}{m^2}.$$

Proof: Using Lemma 5.1 we can write, for $i > 1$,

$$\lim_{k\to\infty} \frac{C^j(i+1)}{kl/m} = \lim_{k\to\infty} \frac{C^j(i+1)m\sqrt{n_k^j(i+1)}}{\sqrt{n_k^j(i+1)}kl} \overset{a.s}{=}$$
$$\lim_{k\to\infty} \beta_{\text{TSP},2}\sqrt{m}\frac{\sqrt{n_k^j(i+1)}}{kl} =$$
$$\lim_{k\to\infty} \frac{\beta_{\text{TSP},2}\sqrt{m}}{kl}\sqrt{\frac{n_k^j(i+1)\frac{kl}{m}C^j(i)}{C^j(i)\frac{kl}{m}}} \overset{a.s.}{=}$$
$$\lim_{k\to\infty} \beta_{\text{TSP},2}\sqrt{\frac{C^j(i)}{kl}}. \tag{3}$$

Let be $X^j(i) \triangleq \lim_{k\to\infty} C^j(i)/kl$; Equation (3) describes a nonlinear recurrence relation for the random variables $X^j(i)$'s: $X^j(i+1) \overset{a.s.}{=} (\beta_{\text{TSP},2}/m)\sqrt{X^j(i)}$. For any sample function such that $X(2) > 0$ (except for a set of probability zero), this recurrence relation converges to the stable equilibrium

$$\lim_{i\to\infty} X^j(i) \overset{a.s.}{=} \frac{\beta_{\text{TSP},2}^2}{m^2}; \tag{4}$$

if, on the other hand, $X(2) = 0$, we trivially have $\lim_{i\to\infty} X^j(i) \overset{a.s.}{=} 0$.

Notice that, for demands arriving in between epochs i and $i+1$ in one of the m subregions, the virtual waiting time is at most $C^j(i) + C^j(i+1)$; therefore, for a demand arriving in between epochs i and $i+1$ in service region j, its maximum virtual waiting time $\tilde{W}^j_{\text{max}}(i)$ satisfies

$$\lim_{i\to\infty}\lim_{k\to\infty} \tilde{W}^j_{\text{max}}(i)/kl \overset{a.s.}{\leqslant} 2\frac{\beta_{\text{TSP},2}^2}{m^2}.$$

The claim follows easily from the definition of $\tilde{W}^j_{\text{max}}(i)$. ■

We finally have the following result for the convergence of expectations.

Theorem 5.3 (from [12]): The following limit holds in heavy load

$$\lim_{\lambda\to\infty}\lim_{r\to\infty} \frac{\mathbb{E}\left[\tilde{W}_r\right]}{\lambda} = \frac{\beta_{\text{TSP},2}^2}{m^2}.$$

From theorem 5.2 and 5.3 we can conclude that, in the heavy load limit, the virtual waiting time increases *linearly* in λ and decreases *quadratically* in m.

The limit results in theorems 5.2 and 5.3 give us a way to estimate, for finite values of λ, an upper bound on the minimum number of agents needed to ensure both condition (i) and condition (ii) in Def. 2. *Assuming* that convergence in theorems 5.2 and 5.3 is well behaved, we can write with increasing accuracy as λ (i.e., kl) $\to \infty$: $\lim_{r\to\infty} \tilde{W}_r \leqslant 2\beta_{\text{TSP},2}^2\lambda/m^2$, and $\lim_{r\to\infty} \mathbb{E}\left[\tilde{W}_r\right] \leqslant \beta_{\text{TSP},2}^2\lambda/m^2$. If we let

$$m_{\text{TSP}} = \sqrt{\frac{2\lambda\beta_{\text{TSP},2}^2}{T_{\text{crit}}}}; \tag{5}$$

then, for any number of vehicles $m \geq \lceil m_{\text{TSP}} \rceil$, the Multiple-Vehicle TSP policy satisfies with arbitrary accuracy both conditions in Def. (2).

Therefore, in heavy load, the minimum number of vehicles m^* satisfies the inequalities

$$\sqrt{\gamma^2\frac{\lambda}{T_{\text{crit}}}} < m^* \leq \sqrt{2\beta_{\text{TSP},2}^2\frac{\lambda}{T_{\text{crit}}}}.$$

Thus, in heavy load, the Multiple-Vehicle TSP policy provides an approximation factor approaching $\sqrt{2\beta_{\text{TSP},2}^2/\gamma^2} \approx 2.6$ on the minimum number of vehicles needed to ensure the desired objective.

VI. Simulation Results for the TSP Policy

In the previous section, we have studied the TSP policy in the heavy load limit and we have found a formula that allows to estimate, for finite values of λ, the minimum number of agents needed by the TSP policy to visit all locations within time T_{crit}. In this section, we show, through simulations, that even for relatively small values of λ the results of the previous section are accurate. Notice that the TSP policy does not perform shortcuts if a demand departs impatiently before service. Clearly in actual applications we would allow shortcuts. In all simulations, we consider the worst case scenario among 100 sample paths. We start with an initial number of demands 10 times greater than the foreseen steady state value. Simulations show that a steady state is reached after about 300 epochs.

We firstly consider a scenario where the demands have an impatience uniformly ditributed in $[0, 90]$ seconds, i.e.:

$$f_L(t) = \begin{cases} 1/90 & \text{if } t \in [0, 90]; \\ 0 & \text{otherwise.} \end{cases}$$

We set an accuracy $\varepsilon = 0.05$; therefore $T_{\text{crit}} = 4.5$ seconds.

We start by studying the length T of the time interval between two epochs in steady state (specifically, we consider the time interval between epochs 1000 and 1001 in service region 1). As discussed above, such time interval is half of the maximum steady state virtual waiting time $\tilde{W}_{\text{max}}$ (i.e., $T = \tilde{W}_{\text{max}}/2$); therefore, we desire $T < T_{\text{crit}}/2$. Consider, in particular, $\lambda = 40$; for such load, formula (5) yields as minimum needed number of agents $m_{TSP} = 4$. As shown in Fig. (2), $m = 4$ is exactly the minimum value capable of ensuring that, in steady state, $T < T_{\text{crit}}/2$. We next investigate the behavior of T for various values of λ; for each λ, we always consider the number of agents dictated by formula (5). From Fig. (3), we see that the simulation results are in good agreement with the analysis carried out in the previous section. Finally, we investigate the performance of the TSP policy for various values of λ; as before, for each λ, we always consider the number of agents dictated by formula (5). From Fig. (4), we notice that the TSP policy always satisfies the requirement that no more than $\varepsilon = 5\%$ of demands depart impatiently. Moreover, we notice some conservatism, as expected from our problem formulation.

Then, we consider an impatience that follows an exponential distribution with the same mean of the previous uniform distribution, i.e.:

$$f_L(t) = \begin{cases} \delta e^{-\delta t} & \text{if } t \geq 0; \\ 0 & \text{otherwise.} \end{cases}$$

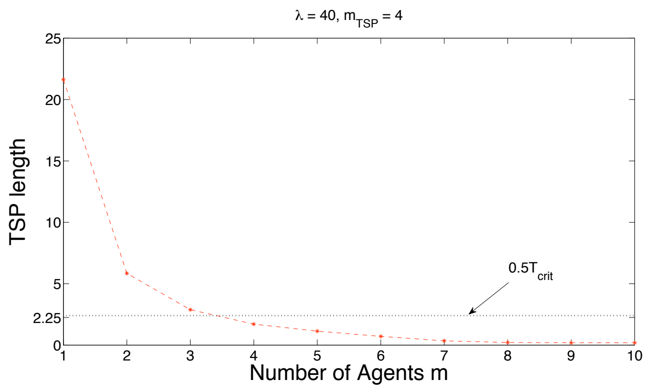

Fig. 2

INTERVAL T AS A FUNCTION OF m; $\lambda = 40$.

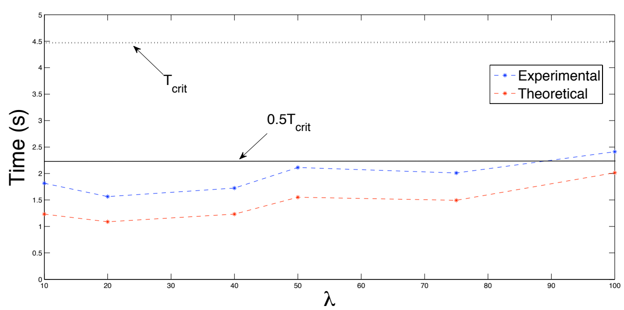

Fig. 3

INTERVAL T AS A FUNCTION OF λ; THE NUMBER OF AGENTS IS RESPECTIVELY $m_{TSP} = 2, 3, 4, 4, 5, 5$.

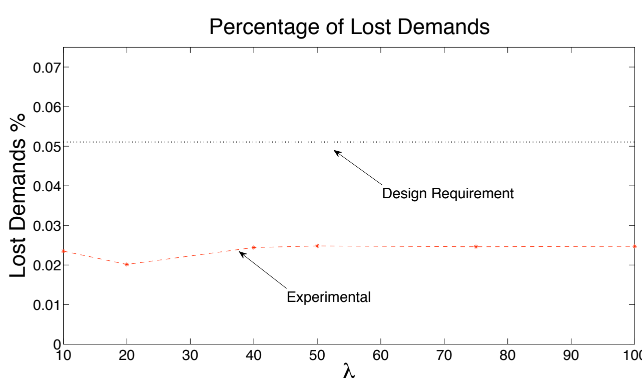

Fig. 4

FRACTION OF DEMANDS THAT DEPART IMPATIENTLY AS A FUNCTION OF λ; THE NUMBER OF AGENTS IS RESPECTIVELY $m_{TSP} = 2, 3, 4, 4, 5, 5$.

where $\delta = 1/45$ seconds. From Fig. (5), we notice that the TSP policy, also in this case, always satisfies the requirement, with a considerable safety margin.

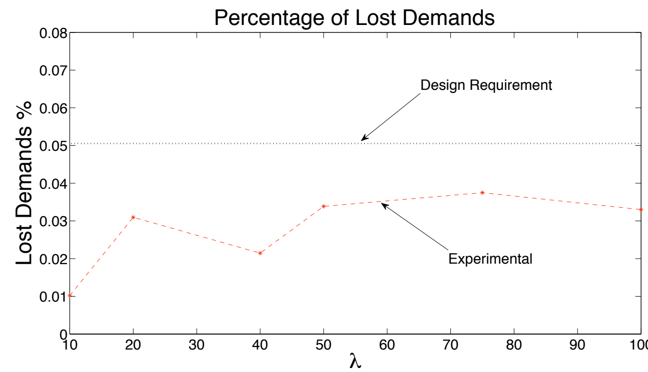

Fig. 5

FRACTION OF DEMANDS THAT DEPART IMPATIENTLY AS A FUNCTION OF λ; THE NUMBER OF AGENTS IS RESPECTIVELY $m_{TSP} = 3, 3, 5, 5, 6, 7$. THE IMPATIENCE TIME FOLLOWS AN EXPONENTIAL DISTRIBUTION.

VII. THE RECEDING HORIZON POLICY: MULTI AGENT CASE

The previous policy is centralized, since it requires a centralized assignment of the service regions. As introduced in [12], assuming that Q is convex, m agents can achieve a configuration with equal service regions in a *decentralized* way.

Toward this end, we introduce certain points, that we call *virtual generators*, associated to each agent; we denote the virtual generators as $G = (g_1, g_2, \dots, g_m) \in \mathbb{R}^{2m}$. We define the region of dominance for agent i as the Voronoi cell $V_i = V(g_i)$; we call the partition into regions of dominance induced by G as $V(G)$. Notice that the virtual generators are not physical points, but are rather artificial variables.

We shall assume that each vehicle has sufficient information available to determine: (1) its Voronoi cell, and (2) the locations of all outstanding events in its Voronoi cell. A control policy that relies on information (1) and (2), is *Voronoi-distributed* in the sense that the behavior of each vehicle depends only on the location of the other agents with contiguous Voronoi cells. Since in a planar Voronoi diagram there are at most $3m - 6$ (m is the number of generators) neighborhood relationships [29, Section 2.3], the number of Voronoi neighbors of each generator is on average less than or equal to 6. (Recall that generators are Voronoi-neighbors if they share an edge, not just a vertex.) Accordingly, Voronoi-distributed policies are *spatially distributed* and *scalable* in the number of agents. A spatially distributed algorithm for the local computation and maintenance of Voronoi cells is provided in [6].

The key idea is to enable the virtual generators to move toward an equitable (i.e., such that regions have the same area) partition $V(G^*)$. Consider the following locational optimization function

$$L_m(G) \triangleq \sum_{i=1}^{m} |V_i|^2.$$

In [12] it is shown that function $L_m(G)$ is non-convex. In what follows, let γ_{ij}, δ_{ij}, O_{ij}^g and O_{ij}^u be defined as in Section II.

Theorem 7.1 (see [12]): The partial derivative of the locational optimization function is:

$$\frac{\partial L_m(G)}{\partial g_i} = 2 \sum_{j \in N(i)} \delta_{ij}\big(|V_i| - |V_j|\big)\Big[\frac{1}{2}n_{ij} + \frac{1}{\gamma_{ij}}(O_{ij}^u - O_{ij}^g)\Big]. \tag{6}$$

where $N(i)$ is the set of indexes of the neighboring regions of V_i. Notice that the computation of the above gradient is decentralized in the sense of Voronoi. Assuming that there exists an equitable Voronoi diagram (henceforth we assume this condition holds, unless otherwise stated), from the expression of the gradient of $L_m(G)$ we see that one of the critical point of $L_m(G)$ will satisfy $|V_i| = |V_j| \quad \forall i, j$. This point correspond to a global minimum of $L_m(G)$, as it can be easily verified by Lagrange multiplier arguments (see [12] for details). We can now state

Theorem 7.2: Assume the virtual generators obey a first order dynamical behavior described by

$$\dot{g}_i = -\frac{\partial L_m(G)}{\partial g_i}. \tag{7}$$

For the closed-loop system induced by equation Eq. (7), the virtual generators converge asymptotically to the set of critical points of $L_m(G)$. In particular, the virtual generators will locally converge to the global minimum of $L_m(G)$, i.e., to an equitable partition of Q.

Remark 7.3: some remarks are in order.

(i) In this section we have postulated that there exists an equitable Voronoi diagram. Although this assumption is reasonable for most workspace's shapes, to date it is not possible to state that for every workspace's shape there always exists a corresponding equitable Voronoi diagram.

(ii) Clearly, the proposed gradient descent law is not guaranteed to find a global minimum of $L_m(G)$. Therefore, the virtual generators will only locally converge to an equitable Voronoi diagram. On the other hand, local optimality is a common price to pay in change of decentralization.

(iii) Since the agents travel inside their own regions of dominance, this policy is inherently safe against collisions.

We finally present the decentralized version of the **Multiple-Vehicle TSP policy**:

Decentralized Multiple-Vehicle TSP policy:

- update own virtual generator according to the updating rule (7);
- while there are no unvisited demands, move at unit speed toward the median of the region of dominance;
- if there are unvisited demands, do the following: (i) compute the TSP tour through all demands in the region of dominance, (ii) service all the demands in such tour. No shortcuts are allowed if a demand departs impatiently before service. Repeat.

VIII. Conclusion

In this paper, we have addressed a stochastic, dynamic multiple-vehicle routing problem, in which demands, associated to points on the plane, are generated over time by a stochastic process, and expire after a random impatience time. Our objective was to determine the minimum number of mobile agents needed to visit each demand before expiration, with probability at least $1-\varepsilon$. In order to attain such objective, we restated the objective in a stronger form, introducing a "critical time" concept, and ensuring that all demands are visited, regardless of their impatience, within such time. This allowed us to compute lower and upper bounds on the minimum number of agents necessary to meet the specifications, and develop decentralized routing algorithms providing a constant-factor approximation to the optimal strategy. Simulations confirm our theoretical results, showing some conservatism. We are currently investigating techniques to remove such conservatism; future work will also include extensions to the non-uniform spatial density case.

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Appendix

In this appendix, we prove Lemma 5.1.

Proof:

Let be $\bar{l} \triangleq l/m$. Consider an arbitrary deterministic time interval c and let $n_k(c)$ be the number of Poisson (with rate $k\bar{l}$) arrivals in a time interval of length c; we start by proving that $\lim_{k\to\infty} n_k(c) \overset{\text{a.s.}}{=} \infty$. From Section II, we have

$$\lim_{k\to\infty} n_k(c) \overset{\text{a.s.}}{=} \infty \Leftrightarrow \forall N > 0 \lim_{k\to\infty} \mathbb{P}\left[\bigcup_{p=k}^{\infty}[n_p(c) < N]\right] = 0.$$

Therefore, we want to show that

$$\forall\, \varepsilon > 0 \quad \exists\, \bar{k} : \forall k > \bar{k} \quad \mathbb{P}\left[\bigcup_{p=k}^{\infty}[n_p(c) < N]\right] < \varepsilon. \tag{8}$$

Now, by using the union bound and assuming $k > k_1 \triangleq \lceil 1/\bar{l}c \rceil$, we have

$$\mathbb{P}\left[\bigcup_{p=k}^{\infty}[n_p(c) < N]\right] \leqslant \sum_{p=k}^{\infty} \mathbb{P}\left[n_p(c) < N\right]$$

$$= \sum_{p=k}^{\infty}\sum_{n=0}^{N-1} e^{-p\bar{l}c}\frac{(p\bar{l}c)^n}{n!} \leqslant N \sum_{p=k}^{\infty} e^{-p\bar{l}c}(p\bar{l}c)^{N-1}.$$

The series $\sum_{p=0}^{\infty} e^{-p\bar{l}c}(p\bar{l}c)^{N-1}$ is convergent (as it can be easily verified with the ratio test); therefore, $\lim_{k\to\infty}\sum_{p=k}^{\infty} e^{-p\bar{l}c}(p\bar{l}c)^{N-1} = 0$. Let k_2 be the smallest integer such that, for all $k > k_2$, $\sum_{p=k}^{\infty} e^{-p\bar{l}c}(p\bar{l}c)^{N-1} < \varepsilon/N$. Then, by letting $\bar{k} = \max(k_1, k_2)$, we prove (8).

Now, the time interval between epoch i and epoch $i+1$ in service region j (call it τ_i^j) is greater than zero almost surely (the only case when $\tau_i^j = 0$ is when all the demands are in the same location of the agent - but this is an event of probability zero). Thus, if Ω is the set of sample functions ω for which both $\tau_i^j > 0$ and $\lim_{k\to\infty} n_k(c) = \infty$, we have $\lim_{k\to\infty} n_k(\tau_i^j) = \infty$ for all ω in Ω. Since $\mathbb{P}[\Omega] = 1$ (and $n_k(\tau_i^j) = n_k^j(i+1)$ by definition), part (i) is proven.

We now prove part (ii). By (1) we have $\lim_{n\to\infty} C^j(i)/\sqrt{n} \overset{a.s.}{=} \beta_{\text{TSP},2}\int_{Q_j}\sqrt{\bar{f}(q)}\,dq$; since in our set-up the spatial density for demands' locations is uniform, the integral $\int_{Q_j}\sqrt{\bar{f}(q)}\,dq$ equals $1/\sqrt{m}$. For any sample function (except possibly for a set of probability zero), $C^j(i)/\sqrt{n_k^j(i)}$ runs through the same sequence of values with increasing k (i.e., λ) as $C^j(i)/\sqrt{n}$ runs through with increasing n. Thus if Ω is the set of sample functions ω for which both $\lim_{n\to\infty} C^j(i)/\sqrt{n} = \beta_{\text{TSP},2}/\sqrt{m}$ and $\lim_{k\to\infty} n_k^j(i) = \infty$, we have $\lim_{k\to\infty} C^j(i)/\sqrt{n_k^j(i)} = \beta_{\text{TSP},2}/\sqrt{m}$. By (1) and part (i) of the lemma we have $\mathbb{P}[\Omega] = 1$, and therefore part (ii) is proven. Part (iii) is an immediate consequence of part (i) and part (ii).

Finally, the number of arrivals in service region j in the time interval between iteration i and $i+1$ is $N_k(C^j(i))$, where $\{N_k(t); t \geqslant 0\}$ is the counting process associated with the Poisson arrival process with intensity $k\bar{l}$. By the strong law of large numbers for renewal processes (see, for example, [30]) we have

$$\lim_{t\to\infty} N_k(t)/t \overset{a.s.}{=} 1/k\bar{l}.$$

Since $\lim_{k\to\infty} C^j(i) \overset{a.s.}{=} \infty$, and with similar arguments as before, we have

$$\lim_{k\to\infty} \frac{N_k(C^j(i))}{C^j(i)\frac{kl}{m}} \overset{a.s.}{=} 1.$$

Since by definition of the policy $n_k^j(i+1) = N_k(C^j(i))$, we get part (iv). ■

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