Maximum-stability dispatch policy for shared autonomous vehicles

Michael W. Levin, Di Kang
Shared autonomous vehicles (SAVs)

- SAV service currently in testing on public roads
- SAVs have safety driver
Motivation

Max-stability dispatch for SAVs

M. W. Levin, D. Kang
SAV: personal vehicle replacement rates

- 1 SAV: 10 personal vehicles
- 1 SAV: 9 personal vehicles
- 1 SAV: 3 personal vehicles

---

Motivation

Max-stability dispatch for SAVs

M. W. Levin, D. Kang

---


Agent-based simulation

Motivation

Max-stability dispatch for SAVs

M. W. Levin, D. Kang

Agent-based simulation

---

---

SAV : personal vehicle replacement rates

- 1 SAV : 10 personal vehicles\(^a\)
- 1 SAV : 9 personal vehicles\(^b\)
- 1 SAV : 3 personal vehicles\(^c\)

---


Queueing model
Define a queue of waiting passengers at each zone: \( w^{rs}(t) \).

Conservation of waiting passengers:

\[
 w^{rs}(t + 1) = w^{rs}(t) + d^{rs}(t) - \min \left\{ \sum_{j \in A} y^{rs}_{rj}(t), w^{rs}(t) \right\}
\]

where \( d^{rs}(t) \) are random variables with mean \( \bar{d}^{rs} \).

\( y^{rs}_{rj}(t) \leq p_r(t) \) is vehicles departing \( r \) for \( s \) to link \( j \).
Define a queue of waiting passengers at each zone: $w^{rs}(t)$.

Conservation of waiting passengers:

$$w^{rs}(t + 1) = w^{rs}(t) + d^{rs}(t) - \min \left\{ \sum_{j \in A} y^{rs}_{rj}(t), w^{rs}(t) \right\}$$

where $d^{rs}(t)$ are random variables with mean $\bar{d}^{rs}$.

- $y^{rs}_{rj}(t) \leq p_{r}(t)$ is vehicles departing $r$ for $s$ to link $j$

This defines a Markov chain on the state space $\mathbb{N}^{|Z|^2}$. 
Vehicle queueing model

- $x_{j}^{rs}(t)$ is the number of vehicles on link $j$ traveling from $r$ to $s$
- $p_{r}(t)$ is the number of vehicles parked at $r$

\[
\sum_{j \in A} \sum_{(r,z) \in \mathbb{Z}^2} x_{j}^{rs}(t) + \sum_{r \in \mathbb{Z}} p_{r}(t) = F
\]
Vehicle queueing model

- $x_j^{rs}(t)$ is the number of vehicles on link $j$ traveling from $r$ to $s$
- $p_r(t)$ is the number of vehicles parked at $r$

$$\sum_{j \in A} \sum_{(r,z) \in Z^2} x_j^{rs}(t) + \sum_{r \in Z} p_r(t) = F$$

Conservation of link queues:

$$x_j^{rs}(t + 1) = x_j^{rs}(t) + \sum_{i \in A} y_{ij}^{rs}(t) - \sum_{j \in A} y_{jk}^{rs}(t) \left| \sum_{k \in \Gamma_j^+} y_{jk}^{rs}(t) \leq x_j^{rs}(t) \right.$$
Vehicle queueing model

- $x_{j}^{rs}(t)$ is the number of vehicles on link $j$ traveling from $r$ to $s$
- $p_{r}(t)$ is the number of vehicles parked at $r$

$$
\sum_{j \in \mathcal{A}} \sum_{(r,z) \in \mathcal{Z}^2} x_{j}^{rs}(t) + \sum_{r \in \mathcal{Z}} p_{r}(t) = F
$$

Conservation of link queues:

$$
x_{j}^{rs}(t+1) = x_{j}^{rs}(t) + \sum_{i \in \mathcal{A}} y_{ij}^{rs}(t) - \sum_{k \in \mathcal{A}} y_{jk}^{rs}(t) \bigg| \sum_{k \in \Gamma_{j}^{+}} y_{jk}^{rs}(t) \leq x_{j}^{rs}(t)
$$

Conservation of parked queues:

$$
p_{r}(t+1) = p_{r}(t) + \sum_{i \in \mathcal{A}} \sum_{q \in \mathcal{Z}} y_{ir}^{qr}(t) - \sum_{j \in \mathcal{A}} \sum_{s \in \mathcal{Z}} y_{rj}^{rs}(t)
$$
Markov decision process

Queues of passengers and vehicles define a Markov chain.

\[
w^{rs}(t + 1) = w^{rs}(t) + d^{rs}(t) - \min \left\{ \sum_{j \in A} y_{rj}^{rs}(t), w^{rs}(t) \right\}
\]

\[
p_{r}(t + 1) = p_{r}(t) + \sum_{i \in A} \sum_{q \in Z} y_{qr}^{ir}(t) - \sum_{j \in A} \sum_{s \in Z} y_{rj}^{rs}(t)
\]

\[
x_{j}^{rs}(t + 1) = x_{j}^{rs}(t) + \sum_{i \in A} y_{ij}^{rs}(t) - \sum_{k \in A} y_{jk}^{rs}(t)
\]

Since vehicle movements can be controlled, this is a Markov decision process model.
Stability and passenger service

\[ \sum_{(r,s) \in \mathbb{Z}^2} w^{rs}(t) \] is the number of waiting passengers at time \( t \)

- If demand is unserved, then \[ \sum_{(r,s) \in \mathbb{Z}^2} w^{rs}(t) \] will increase over time.

---

Queueing model

Max-stability dispatch for SAVs

M. W. Levin, D. Kang
Stability and passenger service

\[ \sum_{(r,s) \in \mathbb{Z}^2} w^{rs}(t) \] is the number of waiting passengers at time \( t \)

- If demand is unserved, then \( \sum_{(r,s) \in \mathbb{Z}^2} w^{rs}(t) \) will increase over time

**Definition**

*The stochastic queueing model is stable if there exists some \( K < \infty \) s.t.*

\[ \frac{1}{T} \sum_{t=1}^{T} \sum_{(r,s) \in \mathbb{Z}^2} \mathbb{E}[w^{rs}(t)] \leq K \quad \forall T \in \mathbb{N} \]

Equivalently, \( \exists \) Lyapunov function \( \nu(w(t)) \geq 0 \) s.t.

\[ \mathbb{E}\left[\nu(w(t+1)) - \nu(w(t))|w(t)\right] \leq \kappa - \epsilon |w(t)| \]

for all \( w(t) \) for \( \kappa < \infty, \epsilon > 0 \).
Assumptions

- SAV travelers wait in the system until served
  - If SAV travelers exited, the concept of stability would need to be redefined.
- Constant travel times for vehicles
- Entire SAV fleet can be centrally dispatched
Maximum-stability policy
Max-pressure policy with maximum stability

\[
\begin{align*}
\text{max} \ & \frac{1}{T} \sum_{\tau=1}^{T} \sum_{(r,s) \in \mathbb{Z}^2} w^{rs}(t) f^{rs}(t + \tau) \\
\text{s.t.} \ & \sum_{s \in \mathbb{Z}} f^{rs}(t + \tau) \leq p_r(t + \tau) \\
\ & p_r(t + \tau + 1) = p_r(t + \tau) + \sum_{q \in \mathbb{Z}} f^{qr}(t + \tau - \Phi^r_q) - \\
& \sum_{s \in \mathbb{Z}} f^{rs}(t + \tau) + \sum_{q \in \mathbb{Z}} \sum_{i \in \mathcal{A}} x_i^{qr}(t + \tau - \Phi^r_i) \\
& f^{rs}(t + \tau) \geq 0
\end{align*}
\]

- \( T \) is the planning horizon — how far we look ahead
- \( f^{rs}(t + \tau) \) anticipates future vehicle dispatch
- \( p_r(t + \tau) \) anticipates future vehicle availability
Planning horizon analysis

\[
\begin{align*}
\text{max} & \quad \frac{1}{T} \sum_{\tau=1}^{T} \sum_{(r,s) \in \mathcal{Z}^2} w^{rs}(t) f^{rs}(t + \tau) \\
\text{s.t.} & \quad \sum_{s \in \mathcal{Z}} f^{rs}(t + \tau) \leq p_r(t + \tau) \\
& \quad p_r(t + \tau + 1) = p_r(t + \tau) + \sum_{q \in \mathcal{Z}} f^{qr}(t + \tau - \Phi^r_q) - \\
& \quad \sum_{s \in \mathcal{Z}} f^{rs}(t + \tau) + \sum_{q \in \mathcal{Z}} \sum_{i \in \mathcal{A}} x^{qr}_i (t + \tau - \Phi^r_i) \\
& \quad f^{rs}(t + \tau) \geq 0
\end{align*}
\]

- \( T \) is the planning horizon — how far we look ahead

\( T \) must be large enough to dispatch vehicles across the network. At least \( \max_r \left\{ \Phi^r_q \right\} \).
Stability region

What demand rates \( \bar{d} \in \mathcal{D} \) could be served by any SAV dispatch policy?

- We want to serve any \( \bar{d} \in \mathcal{D} \) with the max-pressure policy.
Stability region

What demand rates $\bar{d} \in \mathcal{D}$ could be served by any SAV dispatch policy?
- We want to serve any $\bar{d} \in \mathcal{D}$ with the max-pressure policy.

Average SAV flow rates from $r$ to $s$ are enough to serve average demand:

$$\sum_{i \in \Gamma_r^+} \bar{y}_{ri}^s \geq \bar{d}_{rs} \quad \forall (r, s) \in \mathbb{Z}^2$$
Stability region

What demand rates $\bar{d} \in D$ could be served by any SAV dispatch policy?

- We want to serve any $\bar{d} \in D$ with the max-pressure policy.

Average SAV flow rates from $r$ to $s$ are enough to serve average demand:

$$\sum_{i \in \Gamma^+_r} \bar{y}_{ri}^{rs} \geq \bar{d}^{rs} \quad \forall (r, s) \in \mathcal{Z}^2$$

Constraints on average SAV flow rates:

$$\sum_{q \in \mathcal{Z}} \sum_{i \in \Gamma^-_r} \bar{y}_{iq}^{qr} = \sum_{s \in \mathcal{Z}} \sum_{j \in \Gamma^+_r} \bar{y}_{js}^{jr} \quad \forall q \in \mathcal{Z}$$

$$\sum_{i \in \Gamma^-_j} \bar{y}_{ij}^{rs} = \sum_{j \in \Gamma^+_j} \bar{y}_{jk}^{rs} \quad \forall (r, s) \in \mathcal{Z}^2, \forall j \in A_o$$

$$\sum_{(r, s) \in \mathcal{Z}^2} \sum_{(i,j) \in A^2} \bar{y}_{ij}^{rs} \leq F$$
Proposition

If \( \bar{d} \notin \mathcal{D} \), then the system cannot be stabilized by some \( \bar{y} \in \mathcal{Y} \).

Proof. For any SAV dispatch policy \( \exists \) an \( (r, s) \) with an \( \eta > 0 \) s.t.

\[
\sum_{i \in \Gamma^+_r} y_{ri}^{rs} - \bar{d}^{rs} \geq \eta.
\]

Then on average \( w^{rs}(t) \) will increase by \( \eta \) each time step.
Proposition

If $\mathbf{d} \notin \mathcal{D}$, then the system cannot be stabilized by some $\mathbf{y} \in \mathcal{Y}$.

Proof. For any SAV dispatch policy $\exists$ an $(r, s)$ with an $\eta > 0$ s.t.

$$\sum_{i \in \Gamma_r^+} y_{ri}^s - d_{rs}^s \geq \eta.$$ 

Then on average $w^{rs}(t)$ will increase by $\eta$ each time step.

$$\sum_{(r, s) \in \mathcal{E}^2} \sum_{(i, j) \in \mathcal{A}^2} y_{ij}^{rs} \leq F$$

so if $\mathbf{d} \notin \mathcal{D}$, then a larger fleet size is needed to serve $\mathbf{d}$. 
Proposition

The boundary of $\mathcal{D}$ is linear wrt $F$, i.e. if the fleet size increases to $\alpha F$ then demand of $\alpha \bar{d}$ can be stabilized.

Proof. $\alpha F$ admits a linear increase of $\alpha$ in all other constraints defining the stable region.
Proposition

The boundary of $\mathcal{D}$ is linear wrt $F$, i.e. if the fleet size increases to $\alpha F$ then demand of $\alpha \bar{d}$ can be stabilized.

Proof. $\alpha F$ admits a linear increase of $\alpha$ in all other constraints defining the stable region.

An increase in the SAV fleet size should result in a proportional increase in the number of passengers that can be served.
Stability region — maximize service rate

\[
\begin{align*}
\text{max} & \quad \sum_{(r,s) \in \mathcal{Z}^2} \overline{d}^{rs} \\
\text{s.t.} & \quad \sum_{i \in \Gamma_r^+} y_{ri}^{rs} \geq \overline{d}^{rs} \\
& \quad \forall (r, s) \in \mathcal{Z}^2 \\
& \quad \sum_{q \in \mathcal{Z}} \sum_{i \in \Gamma_r^-} y_{iq}^{qr} = \sum_{s \in \mathcal{Z}} \sum_{j \in \Gamma_r^+} y_{jr}^{rs} \\
& \quad \forall q \in \mathcal{Z} \\
& \quad \forall (r, s) \in \mathcal{Z}^2 \\
& \quad \sum_{i \in \Gamma_j^-} y_{ij}^{rs} = \sum_{j \in \Gamma_j^+} y_{jk}^{rs} \\
& \quad \forall (r, s) \in \mathcal{Z}^2, \forall j \in \mathcal{A}_o \\
& \quad \sum_{(r,s) \in \mathcal{Z}^2} \sum_{(i,j) \in \mathcal{A}^2} y_{ij}^{rs} \leq F
\end{align*}
\]

Analytical method to find the theoretical maximum service rate.
Stability region — let $D^0$ be the interior of $D$

Average SAV flow rates from $r$ to $s$ are enough to serve average demand:

$$\sum_{i \in \Gamma^+_r} \bar{y}^{r,s}_{ri} \geq \bar{d}^{r,s} \quad \forall (r, s) \in \mathbb{Z}^2$$

Constraints on average SAV flow rates:

$$\sum_{q \in \mathbb{Z}} \sum_{i \in \Gamma^-_r} \bar{y}^{qr}_{ir} = \sum_{s \in \mathbb{Z}} \sum_{j \in \Gamma^+_r} \bar{y}^{r,s}_{jr} \quad \forall q \in \mathbb{Z}$$

$$\sum_{i \in \Gamma^-_j} \bar{y}^{r,s}_{ij} = \sum_{j \in \Gamma^+_j} \bar{y}^{r,s}_{jk} \quad \forall (r, s) \in \mathbb{Z}^2, \forall j \in A_o$$

$$\sum_{(r,s) \in \mathbb{Z}^2} \sum_{(i,j) \in A^2} \bar{y}^{r,s}_{ij} \leq F$$
Stability proof sketch

If $\bar{d} \in D^0$, then there exists some $\bar{y}$ such that

$$\bar{d}^r_s - \sum_{i \in A} \bar{y}_{ri}^r \leq -\epsilon$$
Stability proof sketch

If $d \in D^0$, then there exists some $\bar{y}$ such that

$$d^{rs} - \sum_{i \in A} y_{ri}^{rs} \leq -\epsilon$$

Proposition

There exists a sequence $(y(t))$ such that

$$\lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T} y(t) = \bar{y}$$

The max-pressure policy constructs a sequence $\hat{y}(t + \tau)$ with limit $\bar{y}$. 
Lemma

Suppose that there exists a $T \in \mathbb{N}$ and a $\kappa_1, \kappa_2 < \infty$ such that

\[
\mathbb{E} [\nu(w(t + T')) - \nu(w(t + T + 1) + \nu(w(t + 1)) - \nu(w(t))|w(t)] \leq \kappa_1
\]
\[
\mathbb{E} [\nu(w(t + T + 1) - \nu(w(t + T))|w(t)] \leq \kappa_2 - \epsilon|w(t)|
\]

then the system is stable.
Lemma

Suppose that there exists a $T \in \mathbb{N}$ and a $\kappa_1, \kappa_2 < \infty$ such that

\[
\mathbb{E} \left[ \nu(w(t + T)) - \nu(w(t + T + 1)) + \nu(w(t + 1)) - \nu(w(t)) \right] \leq \kappa_1
\]

\[
\mathbb{E} \left[ \nu(w(t + T + 1)) - \nu(w(t + T)) \right] \leq \kappa_2 - \epsilon |w(t)|
\]

then the system is stable.

Lemma

Suppose that there exists a $T \in \mathbb{N}$ and a function $\nu(w(t))$ such that

\[
\mathbb{E} \left[ \nu(w(t + T)) - \nu(w(t + T + 1)) + \nu(w(t + 1)) - \nu(w(t)) \right] \leq \kappa_1
\]

\[
\mathbb{E} \left[ \frac{1}{T} \sum_{\tau=1}^{T} (\nu(w(t + \tau + 1)) - \nu(w(t + \tau))) \right] \leq \kappa_2 - \epsilon |w(t)|
\]

then the system is stable.
Lyapunov function

\[ \nu(w(t)) = \sum_{(r,s) \in \mathbb{Z}^2} (w^{rs}(t))^2 \]

**Proposition**

\( \forall \bar{d} \in \mathcal{D}^0 \ \exists M < \infty \text{ such that if } T > M \text{ then the max-pressure control using the planning horizon } T \text{ yields} \)

\[
\mathbb{E} \left[ \frac{1}{T} \sum_{\tau=1}^{T} \sum_{(r,s) \in \mathbb{Z}^2} (w^{rs}(t + \tau + 1))^2 - (w^{rs}(t + \tau))^2 \mid w(t) \right] \leq \kappa - \epsilon |w(t)|
\]

- For any \( \eta > 0 \), there exists a \( M \) s.t. if \( T > M \) then
  \[
  \frac{1}{T} \sum_{\tau=1}^{T} \hat{y}(t + \tau) \leq |\bar{y} - \eta 1|
  \]

- If \( \eta < \epsilon \), \( \exists \epsilon_2 > 0 = \epsilon - \eta \) such that
  \[
  \mathbb{E} \left[ \nu(w(t + 1) - w(t)) \mid w(t) \right] \leq \kappa - \epsilon_2 |w(t)|
  \]
Planning horizon analysis

- For any $\eta > 0$, there exists a $M$ s.t. if $T > M$ then

$$\frac{1}{T} \sum_{\tau=1}^{T} \hat{y}(t + \tau) \leq |\bar{y} - \eta 1|$$

We need $\eta < \epsilon$. 

Stability proof
Max-stability dispatch for SAVs

M. W. Levin, D. Kang
For any \( \eta > 0 \), there exists a \( M \) s.t. if \( T > M \) then
\[
\frac{1}{T} \sum_{\tau=1}^{T} \hat{y}(t + \tau) \leq |\bar{y} - \eta 1|
\]
We need \( \eta < \epsilon \).
\[
\sum_{i \in \Gamma_r^+} \bar{y}_{ri}^s > \bar{d}^r s \Rightarrow \sum_{i \in \Gamma_r^+} \bar{y}_{ri}^s - \bar{d}^r s > \epsilon
\]
For any $\eta > 0$, there exists a $M$ s.t. if $T > M$ then
\[
\frac{1}{T} \sum_{\tau=1}^{T} \hat{y}(t + \tau) \leq |\bar{y} - \eta 1|
\]
We need $\eta < \epsilon$.

\[
\sum_{i \in \Gamma_r^+} \bar{y}_{ri}^s > \bar{d}^r \Rightarrow \sum_{i \in \Gamma_r^+} \bar{y}_{ri}^s - \bar{d}^r > \epsilon
\]

The larger the time horizon, the closer demand can get to the boundary of the stable region.
Numerical results

Sioux Falls
Example — stable demand

![Graph showing the unserved queue over time](image-url)

**Numerical results**

Max-stability dispatch for SAVs
Example — stable demand
Example — unstable demand

![Graph showing unstable demand over time with unserved passengers on the y-axis and time (s) on the x-axis.](image_url)

Numerical results

Max-stability dispatch for SAVs

M. W. Levin, D. Kang
Example — unstable demand

Numerical results
Max-stability dispatch for SAVs

M. W. Levin, D. Kang
Maximum stable demand vs. fleet size

OD demand proportions are constant.

Numerical results

Max-stability dispatch for SAVs

M. W. Levin, D. Kang
Maximum stable demand vs. fleet size

OD demand proportions are constant.
Effect of planning horizon $T$ on maximum stable demand

OD demand proportions are constant.
Conclusions

- Stability analysis of SAVs
- Maximum-stability policy with proof
- Numerical results evaluating stable region

Future work:
- Decentralized policy
- Ridesharing, electric vehicles
- Efficient heuristics, or evaluate stability of heuristic policies
Thank you

Questions?

mlevin@umn.edu