Principles of Robot Autonomy I

Trajectory tracking and closed-loop control
Motion control

• Given a nonholonomic system, how to control its motion from an initial configuration to a final, desired configuration

• Aim
  • Learn how to handle bound constraints via space-time separation
  • Learn about trajectory tracking
  • Learn about closed-loop control

• Readings
Summary of previous lecture

• A nonlinear system $\dot{x} = a(x, u)$ is differentially flat if there exists a set of outputs $z$ such that

$$
x = \beta(z, \dot{z}, \ldots, z^{(q)})
$$

$$
u = \gamma(z, \dot{z}, \ldots, z^{(q)})
$$

• One can then use any interpolation scheme (e.g., polynomial) to plan the trajectory of $z$ in such a way as to satisfy the appropriate boundary conditions

• The evolution of the state variables $x$, together with the associated control inputs $u$, can then be computed algebraically from $z$
Summary of previous lecture

• Constraints on the system can be transformed into the flat output space and (typically) become limits on the curvature or higher order derivative properties of the curve.

• An important class of constraints is represented by bounds on some of the system variables, and in particular the inputs, for example:

\[ |v(t)| \leq v_{\text{max}} \quad \text{and} \quad |\omega(t)| \leq \omega_{\text{max}} \]

• Bound constraints can be effectively addressed via \textit{time scaling}.
Path and time scaling law

• The problem of planning a trajectory can be divided into two steps:
  1. computing a path, that is, a purely geometric description of the sequence of configurations achieved by the robot, and
  2. devising a time scaling law, which specifies the times when those configurations are reached

• Mathematically, a trajectory $\mathbf{x}(t)$ can be broken down into a geometric path $\mathbf{x}(s)$ and a timing law $s = s(t)$, with the parameter $s$ varying between $s(t_0) = s_0$ and $s(t_f) = s_f$ in a monotonic fashion, i.e., with $\dot{s}(t) > 0$

• A possible choice for $s$ is the arc length along the path (in this case, $s_0 = 0$, and $s_f = L$, the length of the path)
Enforcing bound constraints

- Such a space-time separation implies that
  \[ \dot{x}(t) = \frac{dx(t)}{dt} = \frac{dx(s(t))}{ds} \dot{s}(t) \]

- Thus, once the geometric path is determined, the choice of a timing law \( s = s(t) \) will identify a particular trajectory along this path, with a corresponding set of time-scaled inputs (Problem 1 in pset)

- Example, for unicycle model
  - \( v(t) = \frac{d|x(t)|}{dt} = \frac{d|x(s(t))|}{ds} \dot{s}(t) = \tilde{v}(s) \dot{s}(t) \)
  - \( \omega(t) = \frac{d\theta(t)}{dt} = \frac{d\theta(s(t))}{ds} \dot{s}(t) = \tilde{\omega}(s) \dot{s}(t) = \tilde{\omega}(s) \frac{v(t)}{\tilde{v}(s)} \)

- Simplest choice, with \( s \) being arc length: \( s(t) = t \frac{L}{T} \)
Trajectory tracking

• Back to two-step design strategy

$$u^*(t) = u_d(t) + \pi(x(t), x(t) - x_d(t))$$

• Reference trajectory and control history (i.e., $x_d(t)$ and $u_d(t)$) are computed via open-loop techniques (e.g., differential flatness)

• For reference tracking (Problem 3 in pset)
  • Geometric (e.g., pursuit) strategies
  • Linearization (either approximate or exact) + linear structure
  • Non-linear control
  • Optimization-based techniques (e.g., MPC)
Trajectory tracking for differentially flat systems

• Key fact (see, e.g., Levine 2009): a differentially flat system can be linearized by (dynamic) feedback and coordinate change, that is it can be equivalently transformed into the system
  \[ z^{(q+1)} = w \]

• One can then design a tracking controller for the linearized system by using linear control techniques; in particular, for a given reference flat output \( z_d \), define the component-wise error
  \[ e_i := z_i - z_{i,d} \]
  which implies \( e_i^{(q+1)} = w_i - w_{i,d} \)

• For guaranteed convergence to zero of tracking error, one can set
  \[ w_i = w_{i,d} - \sum_{j=0}^{q} k_{i,j} e_i^{(j)} \]
with the gains \( \{k_{i,j}\} \) chosen so as to enforce stability
Trajectory tracking for differentially flat systems

• Example: dynamically extended unicycle model

\[
\begin{align*}
\dot{x}(t) &= V \cos(\theta(t)) \\
\dot{y}(t) &= V \sin(\theta(t)) \\
\dot{V}(t) &= a(t) \\
\dot{\theta}(t) &= \omega(t)
\end{align*}
\]

• The system is differentially flat with flat outputs \((x, y)\), in particular

\[
\begin{bmatrix}
\dot{x}(t) \\
\dot{y}(t)
\end{bmatrix} =
\begin{bmatrix}
\cos(\theta) & -V \sin(\theta) \\
\sin(\theta) & V \cos(\theta)
\end{bmatrix}
\begin{bmatrix}
a \\
\omega
\end{bmatrix} :=
\begin{bmatrix}
w_1 \\
w_2
\end{bmatrix}
\quad := J
\]
Trajectory tracking for differentially flat systems

• Then one can use the following virtual control law for trajectory tracking:

\[ w_1 = \ddot{x}_d + k_{px}(x_d - x) + k_{dx}(\dot{x}_d - \dot{x}) \]
\[ w_2 = \ddot{y}_d + k_{py}(y_d - y) + k_{dy}(\dot{y}_d - \dot{y}) \]

where \( k_{px}, k_{dx}, k_{py}, k_{dy} \geq 0 \) are control gains

• Such a law guarantees exponential convergence to zero of the Cartesian tracking error
Closed-loop control

• General closed-loop control: we want to find

\[ u^*(t) = \pi(x(t), t) \]

• Main techniques:
  • Hamilton–Jacobi–Bellman equation, dynamic programming
  • Lyapunov analysis

For an in-depth study of this topic, see AA203 “Optimal and Learning-based Control” (Spring 2020)
Closed-loop control: posture regulation

• Consider a differential drive mobile robot

\[
\begin{align*}
\dot{x}(t) &= V(t) \cos(\theta(t)) \\
\dot{y}(t) &= V(t) \sin(\theta(t)) \\
\dot{\theta}(t) &= \omega(t)
\end{align*}
\]

• Inputs: \( V \) (linear velocity of the wheel) and \( \omega \) (angular velocity around the vertical axis)

• Goal: drive the robot to the origin \([0, 0, 0]\)
Control based on polar coordinates

• Polar coordinates
  • $\rho$: distance of the reference point of the unicycle from the goal
  • $\alpha$: angle of the pointing vector to the goal w.r.t. the unicycle main axis
  • $\delta$: angle of the same pointing vector w.r.t. the $X_N$ axis

• Coordinate transformation
  • $\rho = \sqrt{x^2 + y^2}$
  • $\alpha = \text{atan2}(y, x) - \theta + \pi$
  • $\delta = \alpha + \theta$
Equations in polar coordinates

• In polar coordinates, the unicycle equations become

\[
\dot{\rho}(t) = -V(t) \cos(\alpha(t)) \\
\dot{\alpha}(t) = V(t) \frac{\sin(\alpha(t))}{\rho(t)} - \omega(t) \\
\dot{\delta}(t) = V(t) \frac{\sin(\alpha(t))}{\rho(t)}
\]

• In order to achieve the goal posture, variables \((\rho, \alpha, \delta)\) should all converge to zero
Control law

• Closed-loop control law (Problem 2 in pset):

\[ V = k_1 \rho \cos(\alpha) \]

\[ \omega = k_2 \alpha + k_1 \frac{\sin(\alpha) \cos(\alpha)}{\alpha} (\alpha + k_3 \delta), \]

• If \( k_1, k_2, k_3 > 0 \), then closed-loop system is globally asymptotically driven to the posture \((0, 0, 0)\)!

Summary

• We covered closed-loop control along two main dimensions
  1. Trajectory tracking (useful to infuse robustness of point-to-point motion)
  2. Posture regulation (useful for final phase of motion)

• We’ll see in Pset 2 how the topics of differential flatness, trajectory tracking, posture regulation, and motion planning will lead to an end-to-end trajectory optimization module
Next time: more on direct / indirect methods

\[ \dot{x}^*(t) = \frac{\partial H}{\partial p}(x^*(t), u^*(t), p^*(t), t) \]

\[ \dot{p}^*(t) = -\frac{\partial H}{\partial x}(x^*(t), u^*(t), p^*(t), t) \]

\[ 0 = \frac{\partial H}{\partial u}(x^*(t), u^*(t), p^*(t), t) \]