# AA274A: Principles of Robot Autonomy I
# Course Notes

September 27, 2019

## 1 Mobile Robot Kinematics

In this section, we discuss robot kinematics, which forms the basis for trajectory planning and control of a robot. Kinematics connects geometry of a robot with time evolution of position, velocity, and acceleration of each of the links in the robot system. A robot's motion is often described in terms of constraints, or a set of equations that the robot needs to obey at all times. Hence, understanding implications of different types of constraints imposed on the robot is a key to robot kinematics. For example, a typical car is free to steer left and right, but it cannot move sideways. Note that this does not mean that cars cannot move laterally; as a matter of fact, a car can reach anywhere in the 2D space with carefully planned trajectories that concatenate motions appropriately. This observation lies at the core of our discussion on robot kinematics. For notational purposes, we first start by how a robot's movement will be described in the general frame.

### 1.1 General Coordinates

A robot's movement is defined by a set of generalized coordinates.

**Definition 1.1** (General Coordinates). *General coordinates refer to a minimum set of coordinates that can specify the unique position of your robot.*

For example, a wheel rolling on a plane can be represented by three parameters, $x, y$, and $\theta$ (Figure 1), where $x, y$ indicates the position at which the wheel touches the ground, and $\theta$ indicates the direction the wheel is traveling in the general frame. This minimum set of parameters $(x, y, \theta)$ are called *configuration variables.*

Robot motion and trajectories are described as time evolution of configuration variables. We use function of time $\xi$ to describe how configuration variables change over time. The trajectory of a wheel rolling on a plane can therefore be described as:

$$\xi(t) : t \to \mathbb{R}^n \tag{1}$$

where $\xi = \begin{bmatrix} x & y & \theta \end{bmatrix}^T$.

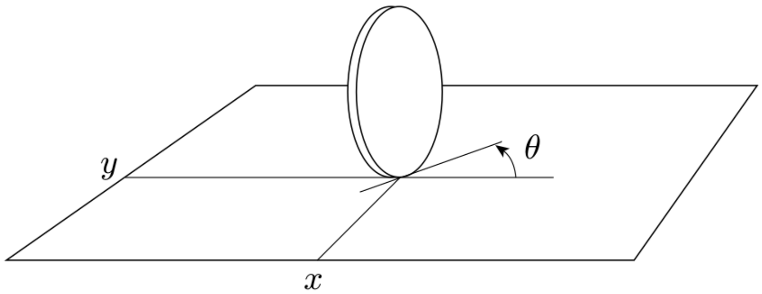


Figure 1: Generalized coordinates for a wheel rolling without slipping on a plane [SSVO08]

## 1.2 Kinematic Constraints

Next we introduce kinematic constraints.

**Definition 1.2** (Kinematic Constraints)**.** *Let the parametric configuration of a robot be denoted as* $\xi = [\xi_1, \ldots, \xi_n]^T$*. Constraints that depend on generalized coordinates and velocities are called kinematic constraints. Equality kinematic constraints can be written as*

$$a_i(\xi, \dot{\xi}) = 0 \qquad i = 1, \ldots, k < n \tag{2}$$

*where* $\dot{\xi} = \frac{d\xi}{dt}$.

Often, kinematic constraints are expressed in *Pfaffian form*, i.e., they are linear in the generalized velocities:

$$\boldsymbol{a}_i^T(\xi)\dot{\xi} = 0 \qquad i = 1, \ldots, k < n \tag{3}$$

or, in matrix form

$$\boldsymbol{A}^T(\xi)\dot{\xi} = \mathbf{0} \tag{4}$$

.

Note that in their most general form, kinematic constraints depend not only on robot's configuration parameters, but also on their derivatives. This is not always the case however. If a kinematic constraint is integrable in the form $h_i(\xi) = 0$, then the constraint is called holonomic. All constraints that cannot be integrated to the form $h_i(\xi) = 0$ are called nonholonomic constraints.

## 1.3 Holonomic Constraints

**Definition 1.3** (Holonomic Constraints)**.** *Constraints that can be put in the form*

$$h_i(\xi) = 0 \qquad i = 1, \ldots, k < n \tag{5}$$

*are called holonomic.*

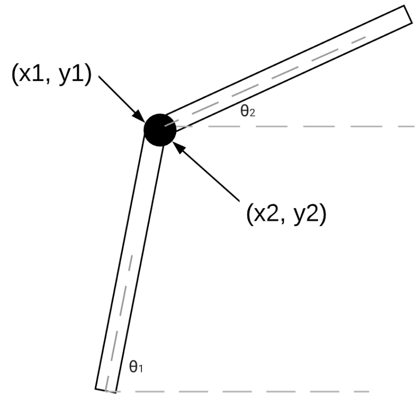


Figure 2: A robot which consists of two rigid body links connected by a joint. Link 1 can be fully described by $x_1, y_1, \theta_1$, and link 2, by $x_2, y_2, \theta_2$. Since the mechanical joint connects the two rigid bodies together, we obtain two holonomic constraints, $x_1 = x_2; y_1 = y_2$. Therefore, even though link 1 and link 2 have 3 degrees of freedom (DOF) individually, we only need 4 configuration variables (6 DOF $(n)-2$ constraints $(k)$) to fully describe kinematics of the robot. This seems natural: the pose of link 1, and the orientation of link 2 are enough to fully describe the robot's pose.

Naturally, the existence of $k$ holonomic constraints implies that of an equal number of kinematic constraints:

$$\frac{dh_i(\xi)}{dt} = \frac{\partial h_i(\xi)}{\partial(\xi)}\dot{\xi} = 0 \qquad i = 1, \ldots, k \tag{6}$$

The converse is often not true, however, as a system of kinematic constraints may or may not be integrable to the form of Eq. 5, as described in the next subsection.

Holonomic constraints often arise from the robot's mechanical interconnections. As described in Figure 2, each holonomic constraint reduces the number of parameters that need to specify a robot's position in the general coordinates by 1. Thus, $k$ holonomic constraints in $n$-dimensional space reduce the space of accessible configurations of the robot to an $n-k$ dimensional subset.

## 1.4 Nonholonomic Constraints

**Definition 1.4** (Nonholonomic Constraints)**.** *Constraints that can be described in Pfaffian form, but cannot be integrated to $h_i(\xi) = 0$ form are called nonholonomic.*

Similar to holonomic constraints, nonholonomic constraints limit the mobility of the mechanical system, but with one important difference: while holonomic constraints reduce accessibility of the robot by limiting the robot configuration to a lower-dimensional subspace, nonholonomic constraints do not limit accessibility at all; instead, they constrain the instantaneous admissible *motion*. This is because Pfaffian form constraints

$$\boldsymbol{a_i}(\xi)\dot{\xi} = 0 \tag{7}$$

allow only velocities from the null space of $\boldsymbol{a}_i(\xi)$. In other words, nonholonomic constraints reduce the set of generalized velocities that can be attained at each configuration, not the set of accessible configuration itself. A mechanical system that is subject to at least one such constraint is called *nonholonomic*.

**Example 1.1** (No-Slip Wheel)**.** We return to the no-slip wheel example to demonstrate a nonholonomic system (Figure 1). The system has no-slip constraint, i.e., the wheel cannot move laterally. This constraint can be described by saying the velocity vector of the wheel does not have any component in the lateral direction. Using the unit vector

$$e_v = (\cos\theta, \sin\theta) \tag{8}$$

to describe the direction in which the wheel is moving, the lateral direction can be expressed as

$$e_{v.\perp} = \Big(\cos(\frac{\pi}{2} - \theta) \;\; - \sin(\frac{\pi}{2} - \theta)\Big) = (\sin\theta, -\cos\theta). \tag{9}$$

by exploiting a simple linear algebraic property, $e_v \cdot e_{v\perp} = 0$.

Since wheel's velocity vector does not have any component in the wheel's lateral direction, the robot's velocity vector is in the null space of $e_{v,\perp}$. Hence, we can express our no-slip constraint as:

$$\begin{bmatrix}\dot{x} & \dot{y}\end{bmatrix} \cdot \begin{bmatrix}\sin\theta \\ -\cos\theta\end{bmatrix} = \dot{x}\sin\theta - \dot{y}\cos\theta = \begin{bmatrix}\sin\theta & -\cos\theta & 0\end{bmatrix}\dot{\xi} = 0 \tag{10}$$

Note that that this is a nonholonomic constraint in Pfaffian form. Since this is a nonholonomic constraint, there is no loss of accessibility, meaning that the wheel can reach anywhere on the plane.

The result shown here can be extended to most wheeled vehicles. In general, wheeled vehicles can nonholonomic system that can be described with general configuration $\begin{bmatrix}x & y & \theta\end{bmatrix}^T$, although a constrained trajectory is necessary to reach a specific goal.

## 1.5 Types of Wheels

The first step to constructing a kinematic model of a wheeled robot is to express constraints on the motions of individual wheels. There are four types of standard wheels with widely varying kinematic properties. While this course note will focus on the modeling of the robot with the fixed standard wheels, more in-depth information can be found in [SNS11].

### 1.5.1 Fixed standard wheel

The **fixed standard wheel** has no vertical axis of rotation for steering. Its angle to the chassis is thus fixed, and it is limited to motion back and forth along the wheel plane and rotation around its contact point with the ground plane. Figure 3 depicts a fixed standard wheel $A$ and indicates its position pose relative to the robot's local reference from$\{X_R, Y_R\}$.

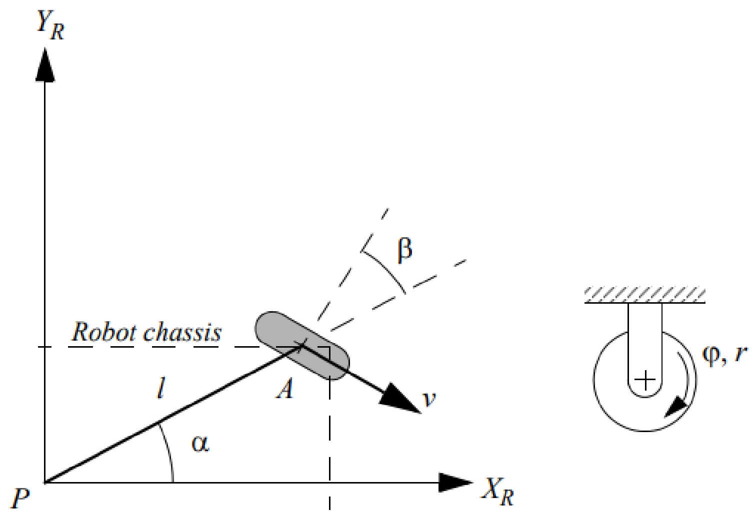


Figure 3: A fixed standard wheel and its parameters [SNS11]

### 1.5.2 Steered standard wheel

The **steered standard wheel** is mostly identical to the fixed standard wheel, except that there is an additional degree of freedom: the wheel may rotate around a vertical axis passing through the center of the wheel and the ground contact point. The equations of motion for the steered standard wheel (Figure 4) are identical to that of the fixed standard wheel, except that the orientation $\beta$ of the wheel to the robot chassis now varies as a function of time.

### 1.5.3 Castor wheel

The **castor wheel** mounts on the robot in an off-centered position and is always fastened to a rotating axis (Figure 5). Since the vertical axis of rotation in a castor wheel does not pass through the ground contact point, zero lateral movement constraint is no longer valid. Unlike in a standard steering wheel, the steering action itself moves the robot chassis thanks to the offset between the ground contact point and the vertical axis of rotation in a castor wheel. Therefore, a robot with only castor wheels can move with any velocity vector in the space of possible robot motions [SNS11].

## 1.6 Kinematic models

Earlier in the chapter we noted that kinematics of a robot describes time evolution of robot dynamics, and such models are described in terms of constraints. In this regard, deriving a model for the whole robot's motion is a bottom-up process. Geometry of each individual wheel describes as much as constraints robot's motion. Wheels are attached to the robot chassis with specific configurations, from which their constraints combine to form constraints

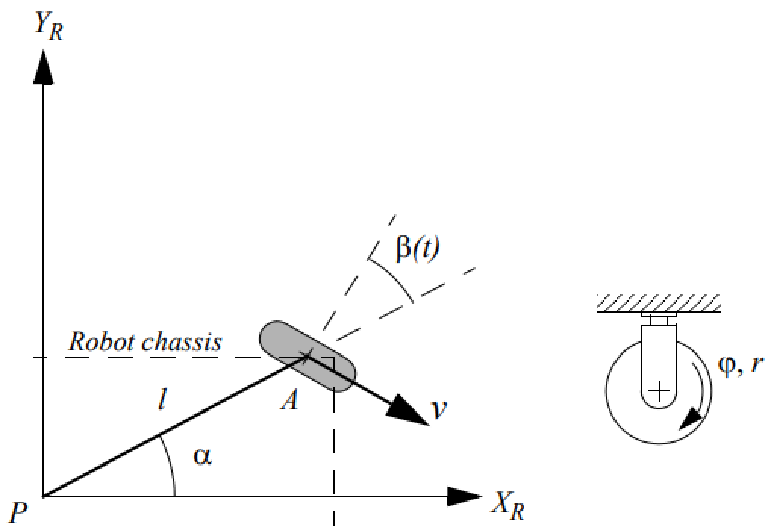


Figure 4: A steered standard wheel and its parameters [SNS11]

on the entire robotic system. Mathematically, this means that the motion of a system subject to $k$ Pfaffian constraints

$$\begin{bmatrix} a_1^T(\xi) \\ \vdots \\ a_k^T(\xi) \end{bmatrix} := A^T(\xi)\dot{\xi} = 0 \tag{11}$$

only allows the admissible velocities at each configuration $\xi$ in $(n-k)$-dimensional null space of matrix $A^T(\xi)$. Admissible trajectories of the robot, therefore, can be characterized as the solutions to

$$\dot{\xi} = \sum_{j=1}^{n-k} g_j(\xi)u_j = G(\xi)u \tag{12}$$

where $u$ indicates the input vector, and $g_1(\xi), \ldots, g_{n-k}(\xi)$ is a basis of the null space of $A^T(\xi)$.

### 1.6.1 Unicycle

A unicycle robot has a single steerable wheel, and its configuration can be fully specified by the parameters $\xi = \begin{bmatrix} x & y & \theta \end{bmatrix}^T$, where $(x, y)$ is the position of the robot and $\theta$ is the orientation of the wheel with respect to the global $x$ axis. The pure rolling (no-slip) constraint for the wheel is:

$$\dot{x}\sin\theta - \dot{y}\cos\theta = \begin{bmatrix} \sin\theta & -\cos\theta & 0 \end{bmatrix} \dot{\xi} = a^T(\xi)\dot{\xi} = 0 \tag{13}$$

Recall that the configuration space is $\xi \in \mathbb{R}^3$. With one kinematic constraint present, we have 2 degrees of mobility, which can be described by two independent bases. We choose a

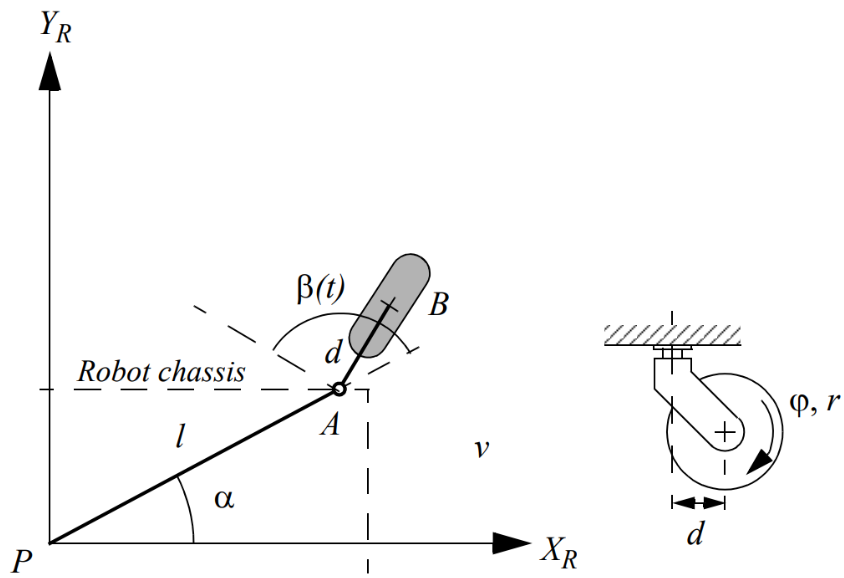


Figure 5: A castor wheel and its parameters [SNS11]

matrix $\boldsymbol{G}$:

$$\boldsymbol{G}(\boldsymbol{q}) = \begin{bmatrix} \boldsymbol{g_1}(\xi) & \boldsymbol{g_2}(\xi) \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix}. \tag{14}$$

where $\boldsymbol{g_1}(\xi)$ and $\boldsymbol{g_2}(\xi)$ are a basis of the null space of the matrix associated with the Pfaffian constraint for each $\xi$. Since any admissible generalized velocities at $\xi$ can be described as a linear combination of these two bases, we arrive at the kinematic model of the unicycle:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \omega \tag{15}$$

where the input $v$ and $\omega$ represent linear and angular velocity, respectively.

### 1.6.2 Differential Drive Model

A differential drive vehicle utilizes two fixed wheels on the rear with a shared axle, and a passive wheel on the front. Figure 7 depicts general coordinates of a differential drive robot: $(x, y)$ is the Cartesian coordinates of the shared axle joining the two wheel centers, and $\theta$ is the orientation of the vehicle chassis with respect to the general X-axis. Differential drive is extremely popular, and Turtlebots in AA274 are also differential drive robots. Fortunately, a differential drive vehicle is kinematically equivalent to a unicycle but mechanically more stable than a unicycle. Intuitively, this makes sense: The no-slip constraints of the two rear wheels are redundant, while the front passive wheel does not impose any constraints on the

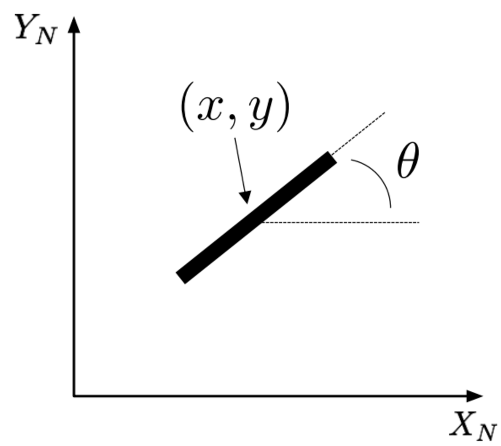


Figure 6: Generalized coordinates for a unicycle [SSVO08]

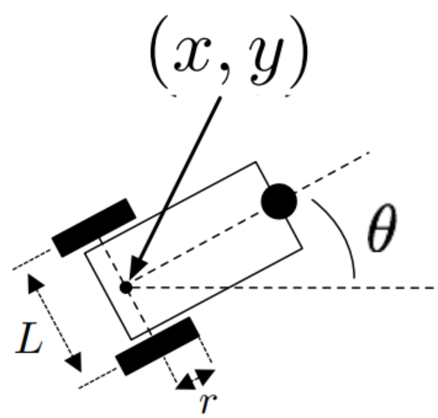


Figure 7: Generalized coordinates for a unicycle [SSVO08]

differential drive robot. As a result, a differential drive robot can be modeled similarly as a unicycle robot with appropriate substitution.

Figure 7 justifies kinematic equivalence of differential drive to a unicycle. Let $\omega$ denotes the angular velocity of the robot; $v_l, v_r$ and $v$ indicates linear velocity of left wheel, right wheel, and the center of the robot respectively. since linear and angular velocity is correlated as

$$v = \omega R \tag{16}$$

and with $L$ denoting the distance between the left and the right wheel, we can rewrite $v_l$

and $v_r$ as

$$v_l = \omega(R - \frac{L}{2}) \tag{17}$$

$$v_r = \omega(R + \frac{L}{2}). \tag{18}$$

And exploiting the geometry, $v_r - v_l = \omega L$, we can write $\omega$ as

$$\omega = \frac{v_r - v_l}{L} \tag{19}$$

By substituting (19) to (16), we obtain an expression for $v$ as well:

$$v = \frac{v_l + v_r}{2} \tag{20}$$

Now we can translate the kinematic model of a unicycle to that of a differential drive robot using (20) and (19).

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{r}{2}(\omega_l + \omega_r)\cos\theta \\ \frac{r}{2}(\omega_l + \omega_r)\sin\theta \\ \frac{r}{L}(\omega_r - \omega_l) \end{bmatrix} \tag{21}$$

## 1.7 Kinematic to Dynamic Models

The kinematic model of a robot forms the basis for planning of robot motions: Based on the kinematic model, the dynamic model of the robot can be derived to specify admissible motions of the robot to the generalized forces. Then the dynamic model of a robot finally lays the foundation for motion control and trajectory planning. In this regard, a kinematic configuration space should be interpreted only as a subsystem of a more general dynamic model [SSVO08]. Since a high fidelity model leads to more reliable motion control, improvements to the kinematic models can be made by placing integrators in front of action variables, or by adding more control variables in the kinematic models. There often exists a trade-off between optimization and fidelity of the model, however, so complexity of kinematic model often varies per applications.

## 1.8 Further Reading

# References


[SNS11] R. Siegwart, I. R. Nourbakhsh, and D. Scaramuzza. *Mobile Robot Kinematics*, pages 57–99. MITP, 2011.

[SSVO08] Bruno Siciliano, Lorenzo Sciavicco, Luigi Villani, and Giuseppe Oriolo. *Robotics: Modelling, Planning and Control (Advanced Textbooks in Control and Signal Processing)*. Springer, 2008.