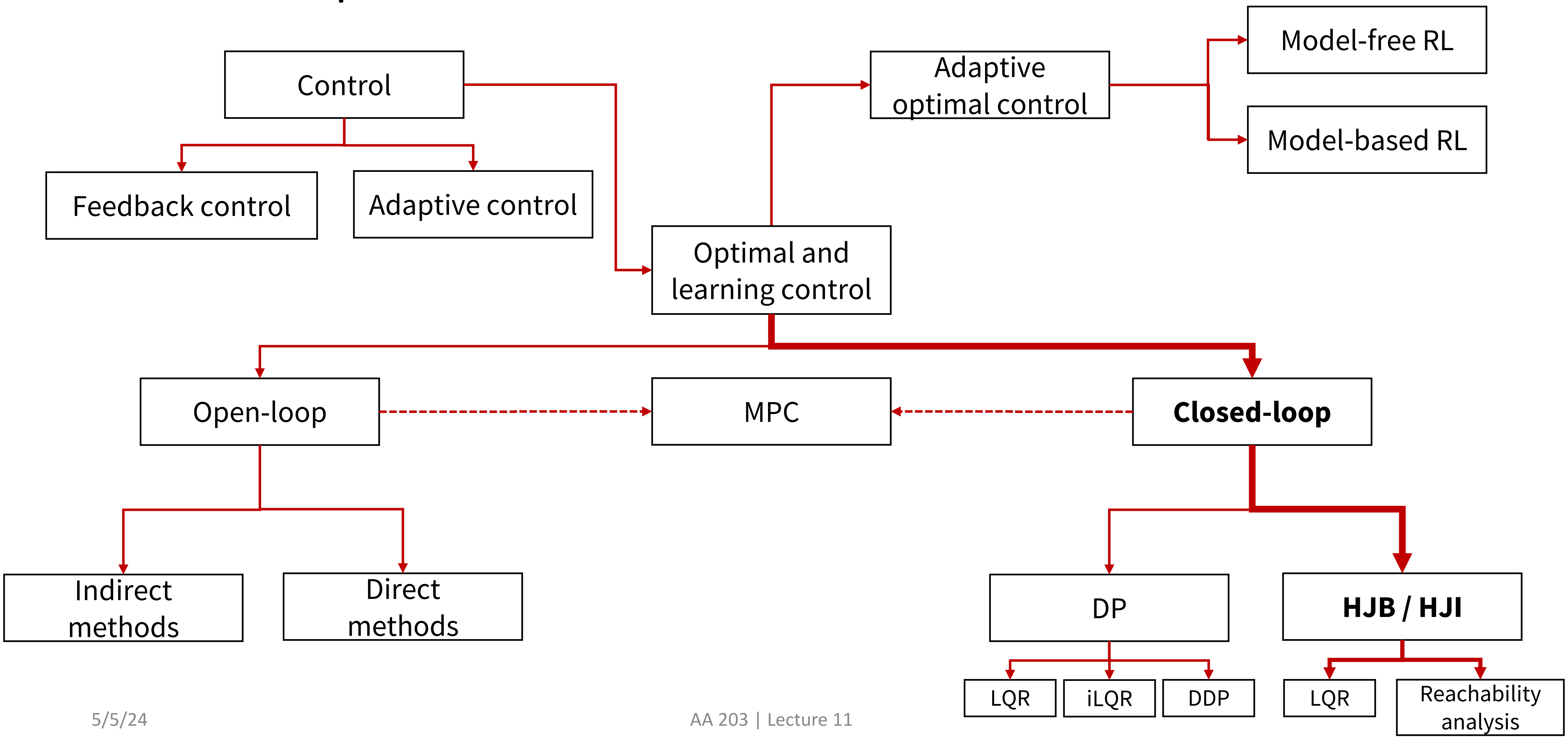
AA203
Optimal and Learning-based Control

HJB, HJI, and reachability analysis

Stanford University

Autonomous Systems Lab

Roadmap



Control
Feedback control
Adaptive control
Optimal and learning control
Adaptive optimal control
Model-free RL
Model-based RL
Open-loop
MPC
Closed-loop
Indirect methods
Direct methods
DP
HJB / HJI
LQR
iLQR
DDP
LQR
Reachability analysis

Dynamic Programming

Previous lectures: focus on discrete-time setting

This lecture: focus on continuous-time setting

- dynamic programming approach leads to HJB / HJI equation: non-linear partial differential equation
- HJB application: solution to continuous LQR problem
- HJI application: reachability analysis

Readings: lecture notes and references therein, in particular:

- Bansal S., Chen M., Herbert S., Tomlin C. J., “Hamilton-Jacobi reachability: A brief overview and recent advances,” 2017.
- Chen M., Tomlin C. J., “Hamilton–Jacobi reachability: Some recent theoretical advances and applications in unmanned airspace management,” 2018.

Continuous-time model

Last time:

- Model: $\mathbf{x}_{k+1} = f(\mathbf{x}_k, \mathbf{u}_k, k)$,
- Cost: $J(\mathbf{x}_0) = h_N(\mathbf{x}_N) + \sum_{k=0}^{N-1} g(\mathbf{x}_k, \mathbf{u}_k, k)$

This time:

- Model: $\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t), t)$,
- Cost: $J\big(\mathbf{x}(t_0)\big) = h\big(\mathbf{x}(t_f), t_f\big) + \int_{t_0}^{t_f} g(\mathbf{x}(\tau), \mathbf{u}(\tau), \tau)\, d\tau$

where t_0 and t_f are fixed

Two-person, zero-sum differential games

What if there is another player (e.g., nature) that interferes with the fulfillment of our objective?

Two-person differential game:

- Model: $\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t), \mathbf{d}(t))$ (*joint* system dynamics),
- Cost: $J\big(\mathbf{x}(t_0)\big) = h\big(\mathbf{x}(0)\big) + \int_{t_0}^{0} g(\mathbf{x}(\tau), \mathbf{u}(\tau), \mathbf{d}(\tau))\, d\tau$
- Player 1, with control $\mathbf{u}(\tau)$, will attempt to *maximize* J, while Player 2, with control $\mathbf{d}(t)$, will aim to *minimize* J, subject to the joint system dynamics
- $\mathbf{x}(\tau)$ is the *joint* system state

Information pattern

- To fully specify the game, we need to specify the *information pattern*
- "Open-loop" strategies
  - Player 1, with control $\mathbf{u}(\tau)$, declares entire plan
  - Player 2, with control $\mathbf{d}(\tau)$, responds optimally
  - Conservative, unrealistic, but computationally cheap

- "Nonanticipative" strategies
  - Other agent acts based on state and control trajectory up to current time
  - Notation: $\mathbf{d}(\cdot) = \Gamma[\mathbf{u}](\cdot)$
  - Disturbance still has the advantage: it gets to (instantaneously) react to the control!

Hamilton-Jacobi-Isaacs (HJI) equation

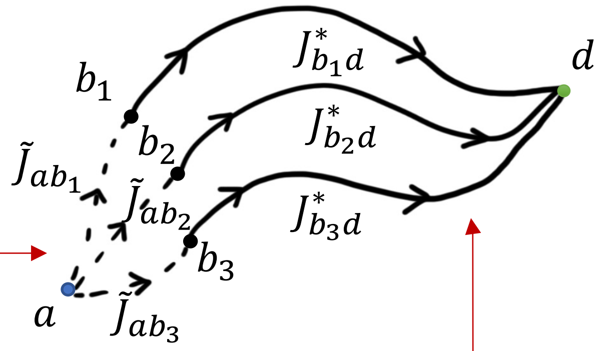
Key idea: apply principle of optimality

The “truncated” problem is

$$J(\mathbf{x}(t), t) = \min_{\Gamma[\mathbf{u}](\cdot)} \max_{\mathbf{u}(\cdot)} \left[\int_t^0 g\big(\mathbf{x}(\tau), \mathbf{u}(\tau), \mathbf{d}(\tau)\big) d\tau + h\big(\mathbf{x}(0)\big) \right]$$

Worst-case disturbance – aims to thwart the controller

HJI equation



- Dynamic programming principle:

$$J(\mathbf{x}(t), t) = \min_{\Gamma[\mathbf{u}](\cdot)} \max_{\mathbf{u}(\cdot)} \left[\int_t^{t+\Delta t} g\big(\mathbf{x}(\tau), \mathbf{u}(\tau), \mathbf{d}(\tau)\big) d\tau + J(\mathbf{x}(t+\Delta t), t+\Delta t) \right]$$

- Approximate integral and Taylor expand $J(\mathbf{x}(t+\Delta t), t+\Delta t)$
- Derive Hamilton-Jacobi-Isaacs partial differential equation (HJI PDE)

HJI equation

- Approximations for small Δt:

$$J(\mathbf{x}(t), t) = \min_{\Gamma[\mathbf{u}](\cdot)} \max_{\mathbf{u}(\cdot)} \left[\int_t^{t+\Delta t} g\big(\mathbf{x}(\tau), \mathbf{u}(\tau), \mathbf{d}(\tau)\big) d\tau + J(\mathbf{x}(t+\Delta t), t+\Delta t) \right]$$

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HJI equation

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- Omit t dependence…

$$J(\mathbf{x}, t) = \max_{\mathbf{u}} \min_{\mathbf{d}} \left[g(\mathbf{x}, \mathbf{u}, \mathbf{d})\Delta t + J(\mathbf{x}, t) + \frac{\partial J}{\partial \mathbf{x}} \cdot \Delta t f(\mathbf{x}, \mathbf{u}, \mathbf{d}) + \frac{\partial J}{\partial t}\Delta t \right]$$

  - Assume (instantaneously) constant $\mathbf{u}$ and $\mathbf{d}$ → optimization over vectors, not functions!
  - Order of max and min reverse (proof given in references)

- $J(\mathbf{x}, t)$ does not depend on $\mathbf{u}$ or $\mathbf{d}$

$$J(\mathbf{x}, t) = J(\mathbf{x}, t) + \max_{\mathbf{u}} \min_{\mathbf{d}} \left[g(\mathbf{x}, \mathbf{u}, \mathbf{d})\Delta t + \frac{\partial J}{\partial \mathbf{x}} \cdot \Delta t f(\mathbf{x}, \mathbf{u}, \mathbf{d}) + \frac{\partial J}{\partial t}\Delta t \right]$$

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HJI equation

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HJI equation

The end result is the Hamilton-Jacobi-Isaacs (HJI) equation

$$0 = \frac{\partial J}{\partial t} + \max_{\mathbf{u}} \min_{\mathbf{d}} \left[g(\mathbf{x}, \mathbf{u}, \mathbf{d}) + \frac{\partial J}{\partial \mathbf{x}} \cdot f(\mathbf{x}, \mathbf{u}, \mathbf{d}) \right]$$

The "Hamiltonian"

with boundary condition

$$J(\mathbf{x}, 0) = h(\mathbf{x})$$

- Given the cost-to-go function, the optimal control for Player 1 is

$$\mathbf{u}^*(\mathbf{x}, t) = \arg\max_{\mathbf{u}} \min_{\mathbf{d}} g(\mathbf{x}, \mathbf{u}, \mathbf{d}) + \frac{\partial J}{\partial \mathbf{x}} \cdot f(\mathbf{x}, \mathbf{u}, \mathbf{d})$$

HJB equation

In case there is no disturbance, end result is the Hamilton-Jacobi-Bellman (HJB) equation

Without a disturbance, $\mathbf{u}$ is usually selected to minimize cost

$$0 = \frac{\partial J}{\partial t} + \min_{\mathbf{u}} \left[g(\mathbf{x}, \mathbf{u}, t) + \frac{\partial J}{\partial \mathbf{x}} \cdot f(\mathbf{x}, \mathbf{u}, t) \right]$$

with boundary condition $J(\mathbf{x}, 0) = h(\mathbf{x})$

- Given the cost-to-go function, the optimal control is

$$\mathbf{u}^*(\mathbf{x}, t) = \arg\min_{\mathbf{u}} g(\mathbf{x}, \mathbf{u}, t) + \frac{\partial J}{\partial \mathbf{x}} \cdot f(\mathbf{x}, \mathbf{u}, t)$$

Continuous-time LQR

Continuous-time LQR: select control inputs to minimize

$$J(\mathbf{x}_0) = \frac{1}{2}\mathbf{x}(t_f)^T H \mathbf{x}(t_f) + \frac{1}{2}\int_{t_0}^{t_f} [\mathbf{x}(t)^T Q(t)\mathbf{x}(t) + \mathbf{u}(t)^T R(t)\mathbf{u}(t)]dt$$

subject to the dynamics

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t)$$

Assumptions:

- $H = H^T \succcurlyeq 0,\ Q(t) = Q(t)^T \succcurlyeq 0, R(t) = R(t)^T \succ 0$
- t_0 and t_f specified
- $\mathbf{x}(t)$ and $\mathbf{u}(t)$ unconstrained

Continuous-time LQR

- As before, value function takes the form: $J(\mathbf{x}(t), t) = \frac{1}{2}\mathbf{x}(t)^T V(t)\mathbf{x}(t)$
- The HJB equation reduces to an ODE (the Riccati equation):

$$-\dot{V}(t) = Q(t) - V(t)B(t)R(t)^{-1}B(t)^T V(t) + V(t)A(t) + A(t)^T V(t)$$

- Riccati equation is integrated backwards, with boundary condition $V(t_f) = H$
- Once we find $V(t)$, the control policy is

$$\mathbf{u}^*(t) = -R(t)^{-1}B(t)^T V(t)\mathbf{x}(t)$$

- Analogously to the discrete case, under some additional assumptions, $V(t) \to$ constant in the infinite horizon setting
- See Notes §3.3 for more details

Applications of differential games

- Pursuit-evasion games
  - homicidal chauffeur problem
  - the lady in the lake

- Reachability analysis

- And many more (e.g., in economics)

Applications of differential games

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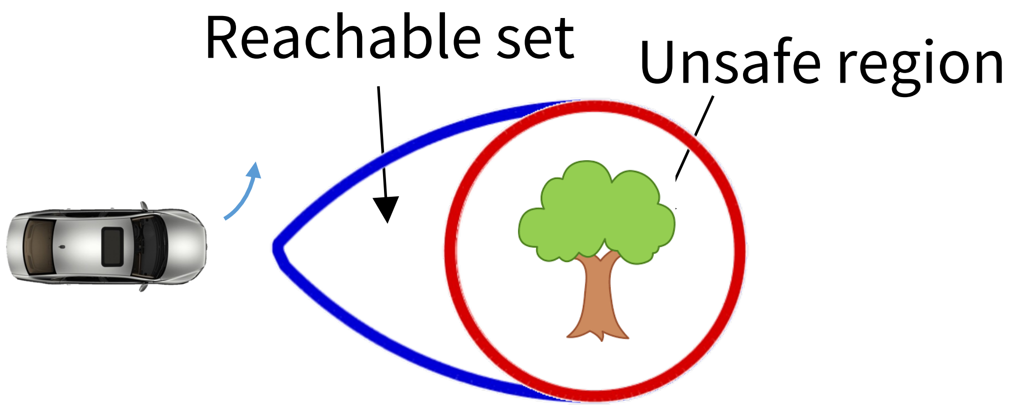
Reachability analysis: avoidance



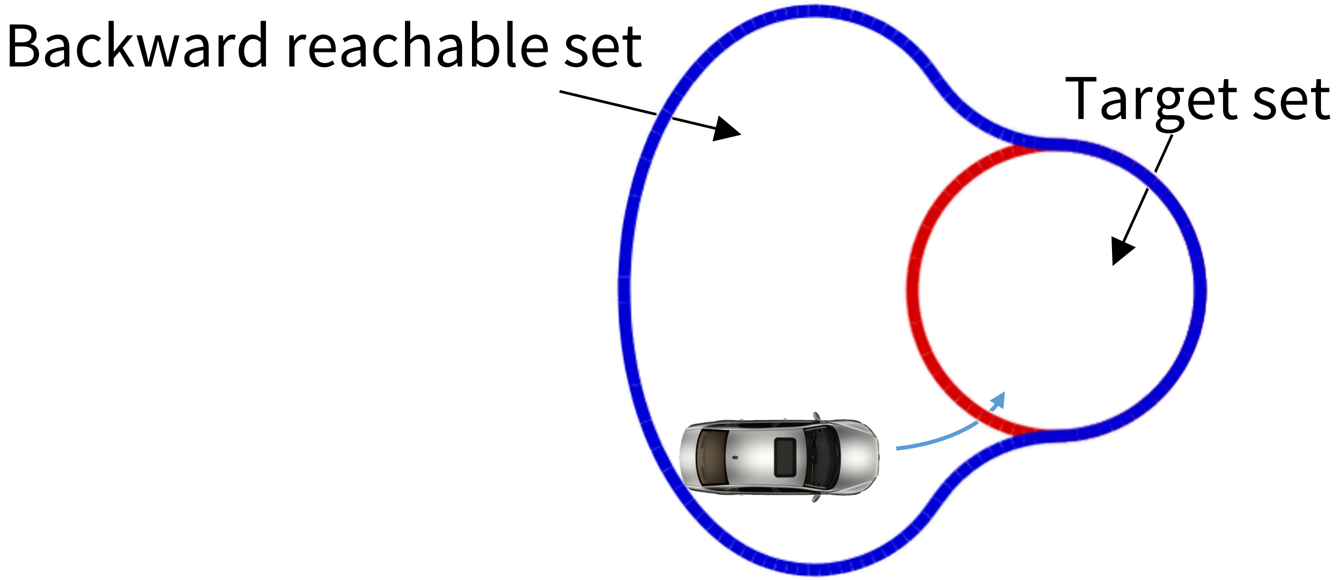
Inputs:
- System model
- Unsafe region: e.g., obstacle

→

Control policy

Backward reachable set
(States leading to danger)

Reachability analysis: goal reaching

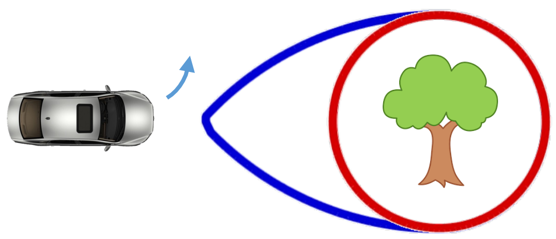


Inputs:
- System model
- Goal region

Control policy

Backward reachable set
(States leading to goal)

Reachability analysis



- $\mathcal{A}(t) = \{\bar{\mathbf{x}}: \exists\Gamma[\mathbf{u}](\cdot), \forall\mathbf{u}(\cdot), \dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, \mathbf{d}), \mathbf{x}(t) = \bar{\mathbf{x}}, \mathbf{x}(0) \in \mathcal{T}\}$

- Model of robot
- Unsafe region

→

Backward reachable set (states leading to danger)

Control policy

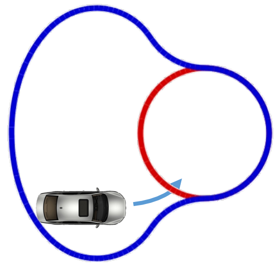
- Model of robot
- Goal region

→

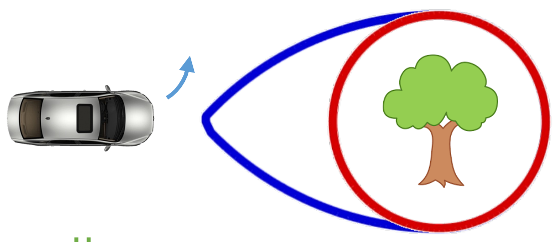
Control policy

Backward reachable set (states leading to goal)

- $\mathcal{R}(t) = \{\bar{\mathbf{x}}: \forall\Gamma[\mathbf{u}](\cdot), \exists\mathbf{u}(\cdot), \dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, \mathbf{d}), \mathbf{x}(t) = \bar{\mathbf{x}}, \mathbf{x}(0) \in \mathcal{T}\}$



Reachability analysis



States at time t satisfying the following:

there exists a disturbance such that for all control, system enters target set at $t = 0$

- $\mathcal{A}(t) = \{\bar{\mathbf{x}} : \exists \Gamma[\mathbf{u}](\cdot), \forall \mathbf{u}(\cdot), \dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, \mathbf{d}), \mathbf{x}(t) = \bar{\mathbf{x}}, \mathbf{x}(0) \in \mathcal{T}\}$

- Model of robot
- Unsafe region

→

Backward reachable set (states leading to danger)

Control policy

- Model of robot
- Goal region

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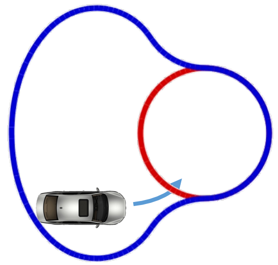
Control policy

Backward reachable set (states leading to goal)



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States at time t satisfying the following:

for all disturbances, there exists a control such that system enters target set at $t = 0$

From HJI to reachability analysis

- Computation of the BRS entails solving a differential game where the outcome is Boolean (the system either reaches the target set or not)
- One can “encode” this Boolean outcome in the HJI PDE by (1) removing the running cost and (2) picking the final cost to denote set membership
  - Value function at each state is the worst case terminal value you can reach

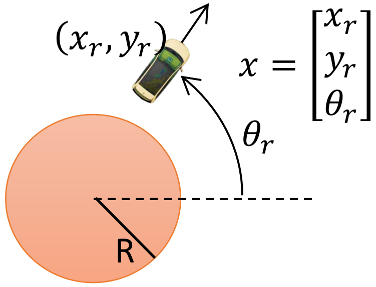
From HJI to reachability analysis

- Hamilton-Jacobi Equation
  - $0 = \frac{\partial J}{\partial t} + \max_{\mathbf{d}} \min_{\mathbf{u}} \left[g(\mathbf{x}, \mathbf{u}, \mathbf{d}) + \frac{\partial J}{\partial \mathbf{x}} \cdot f(\mathbf{x}, \mathbf{u}, \mathbf{d}) \right], \; J(\mathbf{x}, 0) = h(\mathbf{x})$

- Remove running cost
  - $0 = \frac{\partial J}{\partial t} + \max_{\mathbf{d}} \min_{\mathbf{u}} \left[\frac{\partial J}{\partial \mathbf{x}} \cdot f(\mathbf{x}, \mathbf{u}, \mathbf{d}) \right], \; J(\mathbf{x}, 0) = h(\mathbf{x})$

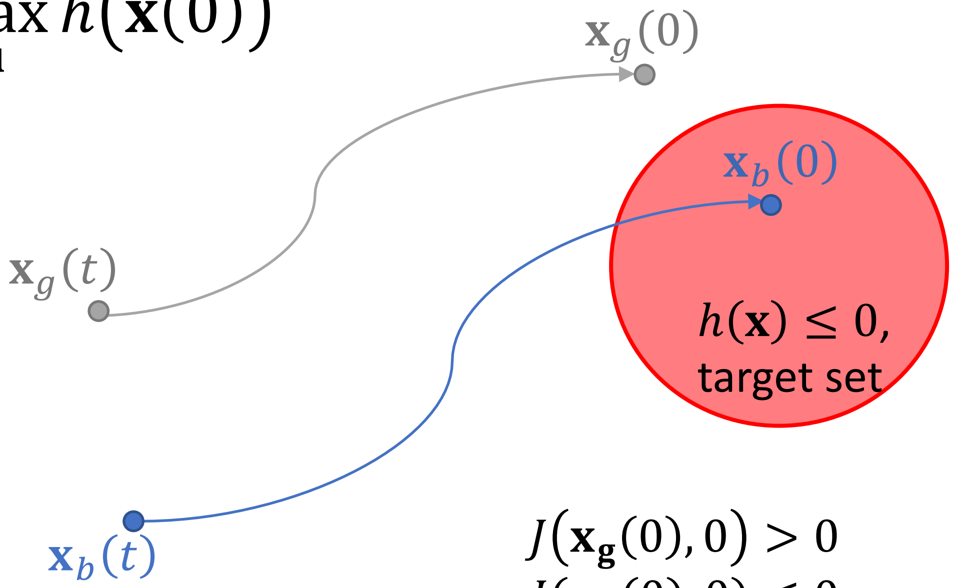
- Pick final cost such that
  - $\mathbf{x} \in \mathcal{T} \Leftrightarrow h(\mathbf{x}) \leq 0$
  - Example: If $\mathcal{T} = \left\{ \mathbf{x} : \sqrt{x_r^2 + y_r^2} \leq R \right\} \subseteq \mathbb{R}^3$, we can pick

$$h(x_r, y_r, \theta_r) = \sqrt{x_r^2 + y_r^2} - R$$



Pick Final Cost

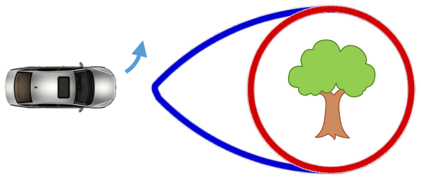
- Why is this correct?
  - Final state $\mathbf{x}(0)$ is in $\mathcal{T}$ if and only if $h\big(\mathbf{x}(0)\big) \leq 0$
  - To avoid $\mathcal{T}$, control should maximize $h\big(\mathbf{x}(0)\big)$
    - Worst-case disturbance would minimize
  - $J(\mathbf{x}, t) = \min_{\Gamma[\mathbf{u}]} \max_{\mathbf{u}} h\big(\mathbf{x}(0)\big)$



$J\big(\mathbf{x_g}(0), 0\big) > 0$
$J\big(\mathbf{x}_b(0), 0\big) \leq 0$

Reaching vs. Avoiding

- Avoiding danger



  - BRS definition

$$\mathcal{A}(t) = \{\bar{\mathbf{x}} : \exists \Gamma[\mathbf{u}](\cdot), \forall \mathbf{u}(\cdot), \dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}, \mathbf{d}), \mathbf{x}(t) = \bar{\mathbf{x}}, \mathbf{x}(0) \in \mathcal{T}\}$$

  - Value function

$$J(\mathbf{x}, t) = \min_{\Gamma[\mathbf{u}]} \max_{\mathbf{u}} h\big(\mathbf{x}(0)\big)$$

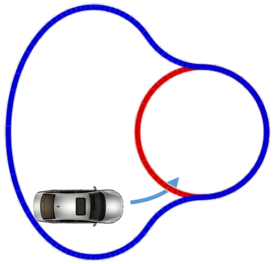
  - HJI

$$\frac{\partial J}{\partial t} + \max_{\mathbf{u}} \min_{\mathbf{d}} \left[\left(\frac{\partial J}{\partial \mathbf{x}}\right)^T f(\mathbf{x}, \mathbf{u}, \mathbf{d}) \right] = 0$$

  - Optimal control

$$\mathbf{u}^* = \arg \max_{\mathbf{u}} \min_{\mathbf{d}} \left(\frac{\partial J}{\partial \mathbf{x}}\right)^T f(\mathbf{x}, \mathbf{u}, \mathbf{d})$$

- Reaching a goal



  - BRS definition

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  - Value function

$$J(\mathbf{x}, t) = \max_{\Gamma[\mathbf{u}]} \min_{\mathbf{u}} h\big(\mathbf{x}(0)\big)$$

  - HJI

$$\frac{\partial J}{\partial t} + \min_{\mathbf{u}} \max_{\mathbf{d}} \left[\left(\frac{\partial J}{\partial \mathbf{x}}\right)^T f(\mathbf{x}, \mathbf{u}, \mathbf{d}) \right] = 0$$

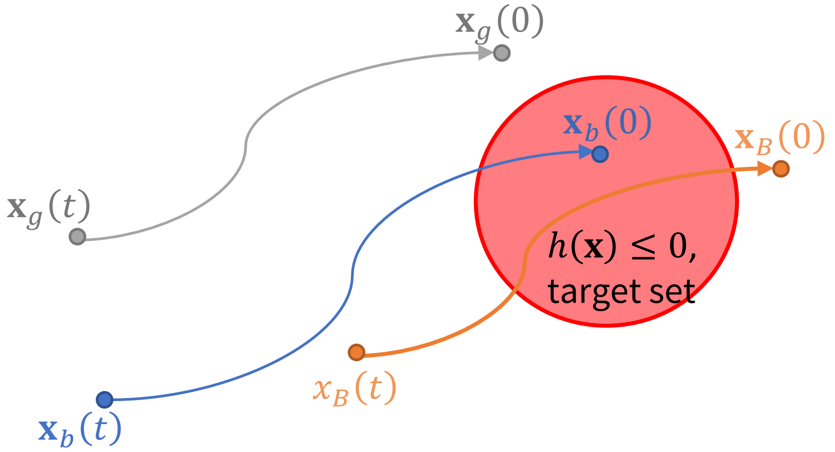
  - Optimal control

$$\mathbf{u}^* = \arg \min_{\mathbf{u}} \max_{\mathbf{d}} \left(\frac{\partial J}{\partial \mathbf{x}}\right)^T f(\mathbf{x}, \mathbf{u}, \mathbf{d})$$

“Sets” vs. “Tubes”

- Backward reachable set (BRS)
  - Only final time matters
  - Initial states that pass through target are not necessarily in BRS
  - Not ideal for safety
- Backward reachable tube (BRT)
  - Keep track of entire time duration
  - Initial states that pass through target are in BRT
  - Used to make safety guarantees

$$J\big(\mathbf{x}_g(0), 0\big) > 0$$
$$J(\mathbf{x}_b(0), 0) \leq 0$$
$$J(\mathbf{x}_B(0), 0) > 0$$

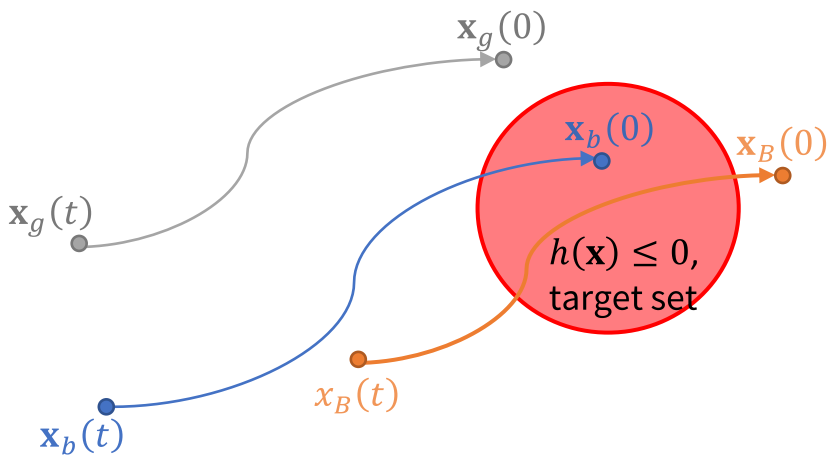


$$J\big(\mathbf{x}_g(0), 0\big) > 0$$
$$J(\mathbf{x}_b(0), 0) \leq 0$$
$$J(\mathbf{x}_B(0), 0) \leq 0$$

"Sets" vs. "Tubes"

- Backward reachable set (BRS)
- Backward reachable tube (BRT)

$J(\mathbf{x}_g(0), 0) > 0$
$J(\mathbf{x}_b(0), 0) \leq 0$
$J(\mathbf{x}_B(0), 0) > 0$



$J(\mathbf{x}_g(0), 0) > 0$
$J(\mathbf{x}_b(0), 0) \leq 0$
$J(\mathbf{x}_B(0), 0) \leq 0$

Value function definition

$$J(\mathbf{x}, t) = \min_{\Gamma[\mathbf{u}]} \max_{\mathbf{u}} h(\mathbf{x}(0))$$

Value function obtained from

$$\frac{\partial J}{\partial t} + \max_{\mathbf{u}} \min_{\mathbf{d}} \left[\left(\frac{\partial J}{\partial \mathbf{x}} \right)^T f(\mathbf{x}, \mathbf{u}, \mathbf{d}) \right] = 0$$

Value function definition

$$J(\mathbf{x}, t) = \min_{\Gamma[\mathbf{u}]} \max_{\mathbf{u}} \min_{\tau \in [t, 0]} h(\mathbf{x}(\tau))$$

Value function obtained from

$$\frac{\partial J}{\partial t} + \min \left\{ \max_{\mathbf{u}} \min_{\mathbf{d}} \left[\left(\frac{\partial J}{\partial \mathbf{x}} \right)^T f(\mathbf{x}, \mathbf{u}, \mathbf{d}) \right], 0 \right\} = 0$$

Computational aspects

- Computational complexity (traditional PDE solver)
  - $J(\mathbf{x}, t)$ is computed on an $(n + 1)$-dimensional grid
  - $n \leq 5$ is reasonable; larger requires some compromises
  - Dimensionality reduction methods (decoupling) sometimes help

- Alternatives/related approaches
  - Sacrifice global optimality
  - Give up guarantees

  - NN-based PDE solvers
  - Sampling-based methods
  - Reinforcement learning

Example: pursuit/evasion with two identical vehicles

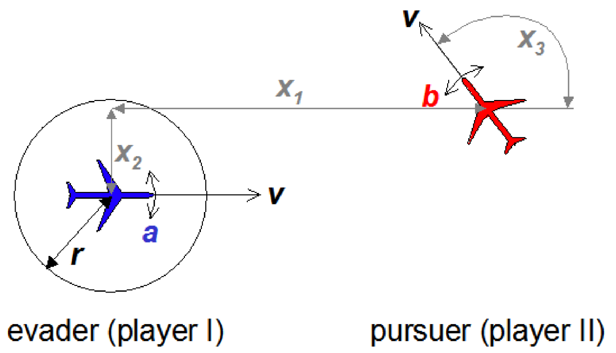
- With evader (a), pursuer (b) dynamics

$$\begin{bmatrix} \dot{x}_a \\ \dot{y}_a \\ \dot{\theta}_a \end{bmatrix} = \begin{bmatrix} v\cos(\theta_a) \\ v\sin(\theta_a) \\ u_a \end{bmatrix}, \quad \begin{bmatrix} \dot{x}_b \\ \dot{y}_b \\ \dot{\theta}_b \end{bmatrix} = \begin{bmatrix} v\cos(\theta_b) \\ v\sin(\theta_b) \\ u_b \end{bmatrix}, \quad u_a, u_b \in [-u_{\max}, u_{\max}]$$

we consider the relative system in (a)'s frame

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -v + v\cos(x_3) + u_a x_2 \\ v\sin(x_3) - u_a x_1 \\ u_b - u_a \end{bmatrix}$$

Courtesy of [Ian Mitchell, "ToolboxLS"](), Section 2.6.1



Next time

- Model Predictive Control