Convex Optimization & Optimization Tools

AA 203 Recitation #2

April 12th, 2024

Agenda

Preliminaries

- Why study Convex Optimization?
- Convex Sets & Convex Functions
- Convex Programming

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Examples of Convex Optimization

- Linear Programming and Duality
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CVXPY: Convex Optimization in Python

- Least Squares
- Discrete LQR

Preliminaries

Optimization

Optimization problems typically take the following form:

$$\text{minimize } f(x)$$
$$\text{subject to } x \in S,$$

where $f : S \to \mathbb{R}$ is a function and S is some some set that can generally be described by the intersection of equality and inequality constraints

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Convex Optimization imposes a special structure of "convexity" on both the function f and the constraint set S

Why study Convex Optimization?

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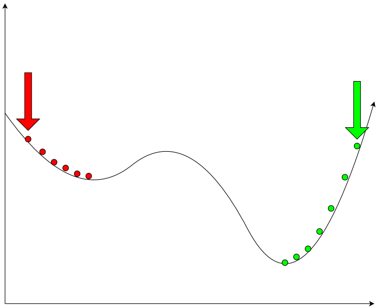

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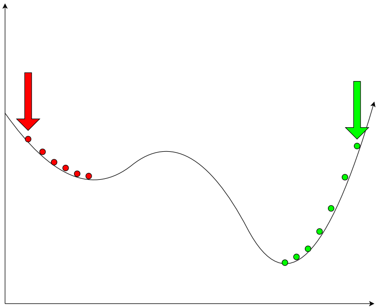

Observation 3: Under non-convexities it is often computationally hard to find global minimizers.

Convex Functions

Definition (Convex Functions)

A function $f : S \to \mathbb{R}$ is convex if for any $x_1, x_2 \in S$ and any $\alpha \in [0, 1]$, it holds that

$$f(\alpha x_1 + (1 - \alpha)x_2) \leq \alpha f(x_1) + (1 - \alpha)f(x_2).$$

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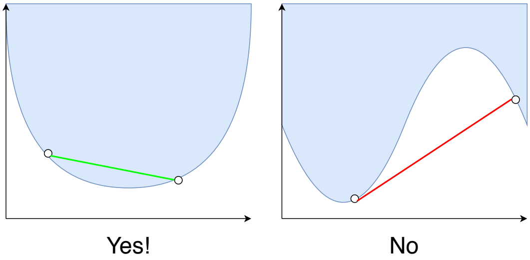

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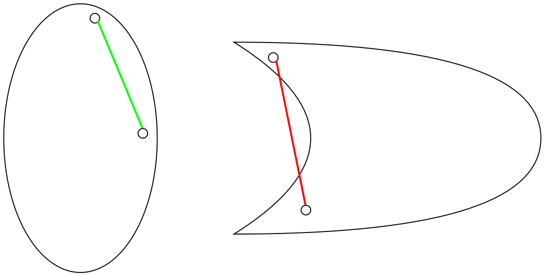

Yes! No

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A convex program (aka convex optimization problem) is a minimization problem of a convex function over a convex set:

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where S is a convex set and $f : S \to \mathbb{R}$ is a convex function.

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Suppose a set S is described by the intersection of equality and inequality constraints

$$g_i(x) \leq 0, \text{ for } i = 1, \ldots, m,$$
$$h_j(x) = 0, \text{ for } j = 1, \ldots, k.$$

Then, S is convex if the functions $h_j(x)$ are linear, and the functions $g_i(x)$ are convex.

Recipe to Identify Convex Programs

An optimization problem

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is convex if

1. The function $f(x)$ is convex
2. The functions $h_j(x)$ are linear
3. The functions $g_i(x)$ are convex

Examples

Is the following problem convex?

$$
\begin{aligned}
\text{minimize } & c^T x \\
\text{subject to } & a_i^T x \leq 0, \ \text{for } i = 1, \dots, m, \\
& b_j^T x = 0, \ \text{for } j = 1, \dots, k.
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This problem is not convex, since the equality constraint is non-linear. But it can be convexified as:

$$\begin{aligned} &\text{minimize } c^T x \\ &\text{subject to } \ ||x||^2 \leq 1. \end{aligned}$$

Convex Program: Local Optima are Global Optima

Definition (Local Minimum)

For an optimization problem $\min_{x \in S} f(x)$, a point x^* is a local minimum if there exists some $\epsilon > 0$ so that for every $x \in S$ with $||x - x^*||_2 \leq \epsilon$, $f(x^*) \leq f(x)$.

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Theorem (Equivalence of Local and Global Optima)

Let $\min_{x \in S} f(x)$ *be a convex program. If* x^* *is a local minimum, then* $f(x^*) \leq f(x)$ *for every* $x \in S$*. In other words,* x^* *is a global minimum.*

Convex Program: Local Optima are Global Optima

Proof: (by contradiction) Suppose x^* is a local but not global minimum.

Since x^* is a local optima, there exists $\epsilon > 0$ so that $f(x^*) \leq f(x)$ for all $x \in S$, $||x - x^*||_2 \leq \epsilon$.

Since x^* is not a global minimum, we can find $x_0 \in S$ where $f(x_0) < f(x^*)$.

Since S is convex, $\alpha x^* + (1-\alpha)x_0 \in S$ for every $\alpha \in [0,1]$.

Note that $f((1-\alpha)x^* + \alpha x_0) \leq (1-\alpha)f(x^*) + \alpha f(x_0) < f(x^*)$.

Pick $\alpha' = \frac{\epsilon}{2||x^*-x_0||_2}$ and set $x' := (1-\alpha')x^* + \alpha' x_0$.

We have $f(x') < f(x^*)$ and $||x^* - x'||_2 \leq \epsilon$.

This contradicts the fact that x^* is a local minimum. □

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S not convex examples: Optimal Control of Nonlinear Systems, Integer Programming.

f not convex examples: Training Neural Networks.

Examples of Convex Optimization

Optimization Models and Tools

We will focus on two of the most common convex Optimization Examples:

1. Linear Programming (LP) and Duality
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- Convex Programming (CP).
- Mixed-Integer Linear Programming (IP).

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Optimization Software

- CVXPY (LP, QP, SDP, CP, IP).
- CPLEX (LP, QP, IP).

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A linear programming instance is specified by
$c \in \mathbb{R}^n$, $b \in \mathbb{R}^p$, $A \in \mathbb{R}^{p \times n}$, $b_{eq} \in \mathbb{R}^q$, $A_{eq} \in \mathbb{R}^{q \times n}$.

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Software (CVXPY):

```
x = cvx.Variable(n)
prob = cvx.Problem(cvx.Minimize(c.T@x), [A @ x <= b])
prob.solve()
```

LP Duality

Suppose we have the following "Primal" linear program:

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{minimize}}\ & c^T x \\ \text{subject to}\ & Ax \leq b, \\ & x \geq 0. \end{aligned}$$

Then, it has the following dual

$$\begin{aligned} \underset{x\in\mathbb{R}^n}{\text{maximize}}\ & b^T y \\ \text{subject to}\ & A^T y \geq -c, \\ & y \geq 0. \end{aligned}$$

Why is Duality Important?

Weak Duality: The optimal objective value of the dual problem is always a lower bound on the optimal objective value of the primal problem, i.e., $c^T x^* \geq b^T y^*$.

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Strong Duality: If the primal problem has a feasible solution, then the optimal objective value of the dual problem is exactly equal to the optimal objective value of the primal problem, i.e., $c^T x^* = b^T y^*$.

Shadow Price Interpretation: The dual variables of the constraints of the primal problem can be interpreted as prices.

LP Example - Resource Allocation

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(2) ensures that no good is sold more than its capacity. (3) ensures that no user gets more than one good.

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Let p be the prices in the market. Then, each person t_i wishes to maximize their payoff given by

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That is, users wish to purchase any good j such that $j \in \arg\max_{j \in [m]} \{u_{ij} - p_j\}$ as long as $u_{ij} \geq p_j$ for some j.

LP Example - Resource Allocation

Let p_j be the dual of the capacity constraints and λ_i be the dual of the allocation constraints. Then, we have the following dual problem:

$$\begin{aligned}
\underset{p \in \mathbb{R}^m, \lambda \in \mathbb{R}^n}{\text{minimize}} \; & \sum_{j=1}^{m} p_j b_j + \sum_{i=1}^{n} \lambda_i \\
\text{subject to } & \lambda_i \geq u_{ij} - p_j \text{ for all } 1 \leq i \leq n, 1 \leq j \leq m \\
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LP Duality gives a method to set prices and achieve a decentralized implementation of the optimal solution.

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As a consequence, all points in S can be written as convex combinations of the extreme points of S.

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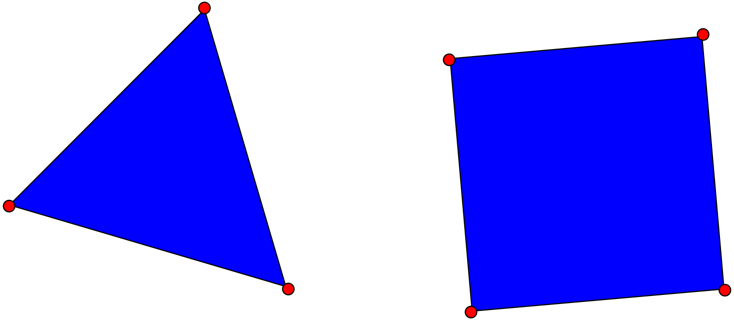

Linear Programming - Properties

Theorem (Extreme Solutions of Linear Programs)

If a linear program $\min_{x \in P} c^\top x$ has a finite optimal value (i.e. it has a non-empty solution set), then the solution set contains at least one extreme point of P.

Linear Programming - Properties

Theorem (Extreme Solutions of Linear Programs)

If a linear program $\min_{x \in P} c^\top x$ *has a finite optimal value (i.e. it has a non-empty solution set), then the solution set contains at least one extreme point of* P*.*

Proof: Let $x^* \in P$ be an optimal solution.

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So there is some $x' \in E_P$ with $c^\top x' \leq c^\top x^*$.

Since x^* is a minimizer, x' must also be a minimizer.

Quadratic Programming

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A quadratic programming instance is specified by
$f \in \mathbb{R}^n, H \in \mathbb{R}^{n \times n}, b \in \mathbb{R}^p, A \in \mathbb{R}^{p \times n}, b_{eq} \in \mathbb{R}^q, A_{eq} \in \mathbb{R}^{q \times n}$.

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Software (CVXPY):

```
x = cvx.Variable(n)
prob = cvx.Problem(cvx.Minimize((1/2) * cvx.quad_form(x, H) + f.T @ x), [A
@ x <= b, A_eq@x == b_eq])
prob.solve()
```

QP Example: Discrete LQR

Given a discrete linear dynamical system

$$x_{t+1} = Ax_t + Bu_t$$

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\text{subject to } & x_{t+1} = A x_t + B u_t \text{ for all } 0 \le t \le T-1 \quad (4) \\
& x_0 = \text{initial condition} \quad (5) \\
& \quad (6)
\end{aligned}
$$

CVXPY: Convex Optimization in Python

Problem Objects in CVXPY

Instantiate by specifying an objective function and constraints.

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prob = cvx.Problem(objective, constraints)
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The objective value of the solution can be found at `prob.value`

Least Squares in CVXPY

Recall the Least squares problem:

$$\min_{x \in \mathbb{R}^m} ||Ax - b||_2^2$$

where $A \in \mathbb{R}^{n \times m}$, $b \in \mathbb{R}^n$.

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Problem setup

```
import numpy as np
import cvxpy as cvx

n = 10
m = 5

A = np.random.normal(0,1,(n,m))
b = np.random.normal(0,1,(n,))
```

Least Squares in CVXPY

Solving the problem

```
x = cvx.Variable(m)

objective = cvx.Minimize(cvx.sum_squares(A @ x - b))
constraints = []

prob = cvx.Problem(objective, constraints)
prob.solve()

print(prob.status)
print(prob.value) # optimal objective value
print(x.value) # get the optimal solution
```

Discrete LQR in CVXPY

Recall the Discrete LQR problem:

$$
\begin{aligned}
\underset{u \in \mathbb{R}^T}{\text{minimize}} \ & \frac{1}{2} x_T^\top Q_T x_T + \frac{1}{2} \sum_{t=0}^{T-1} x_t^\top Q x_t + u_t^\top R u_t \\
\text{subject to } & x_{t+1} = A x_t + B u_t \text{ for all } 0 \leq t \leq T-1 \\
& x_0 = \text{initial condition}
\end{aligned}
$$

Discrete LQR in CVXPY

Problem setup

```
import numpy as np
import cvxpy as cvx

n = 5 # state dimension (x)
m = 5 # control dimenion (u)
T = 20 # number of timesteps in planning horizon
u_bound = 1.0 # bound on control effort

Q = np.eye(n) # state deviation cost
R = 2*np.eye(m) # control effort cost
A = np.random.normal(0,1,(n,n)) # dynamics
B = np.random.normal(0,1,(n,m))

x_0 = np.random.normal(0,1,(n,)) # initial condition
```

Discrete LQR in CVXPY

Iterative building of objective and constraints

```
X = {}
U = {}
cost_terms = []
constraints = []
```

Discrete LQR in CVXPY

Iterative building of objective and constraints

```
for t in range(T):
    X[t] = cvx.Variable(n) # state variable for time t
    U[t] = cvx.Variable(m) # control variable for time t
    cost_terms.append( cvx.quad_form(X[t],Q) ) # state cost
    cost_terms.append( cvx.quad_form(U[t],R) ) # control cost

    if (t == 0):
        constraints.append( X[t] == x_0) # initial condition

    if (t < T-1 and t > 0):
        # dynamics constraint
        constraints.append( A @ X[t-1] + B @ U[t-1] == X[t] )
```

Discrete LQR in CVXPY

Solving the Problem

```
objective = cvx.Minimize(cvx.sum(cost_terms))

prob = cvx.Problem(objective, constraints)
prob.solve()
print(prob.status) # optimal, infeasible, etc.
print(prob.value) # optimal objective value
print(U[0].value) # optimal control
```

Key Takeaways

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6. Quadratic Programming
7. Basic Implementation on CVXPY