AA203 Optimal and Learning-based Control

Lecture 12

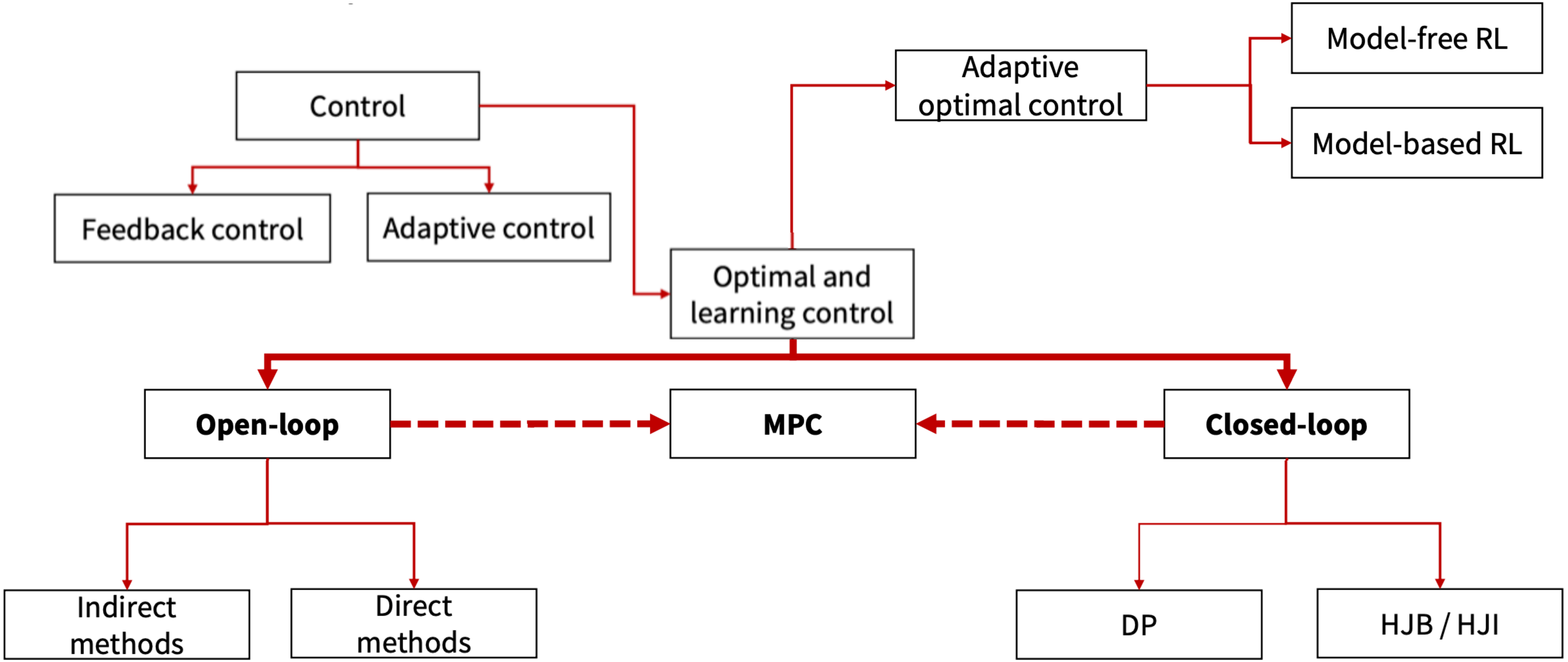
Introduction to Model Predictive Control; Stability

Autonomous Systems Laboratory
Daniele Gammelli

Stanford University

Autonomous Systems Laboratory
Stanford Aeronautics & Astronautics

Roadmap



Control
Feedback control
Adaptive control
Optimal and learning control
Adaptive optimal control
Model-free RL
Model-based RL
Open-loop
MPC
Closed-loop
Indirect methods
Direct methods
DP
HJB / HJI

Outline of the next two lectures

MPC: Basic setting and key ideas

Main design choices:

- Persistent feasibility
- Stability

Implementation aspects of MPC

Further reading:

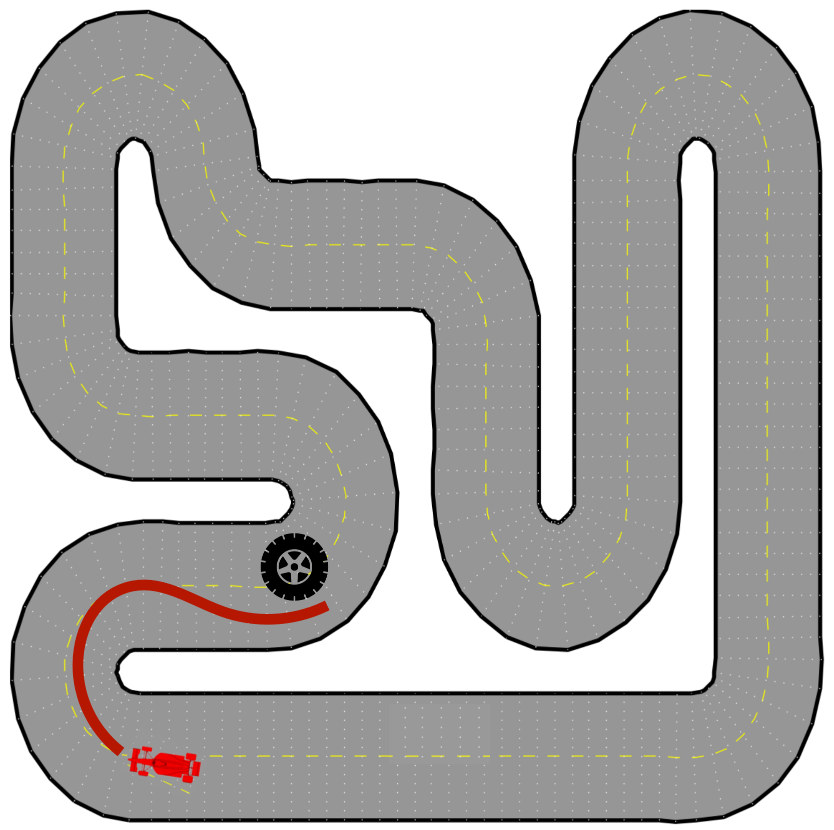
- F. Borrelli, A. Bemporad, M. Morari. *Predictive Control for Linear and Hybrid Systems*, 2017.
- J. B. Rawlings, D. Q. Mayne, M. M. Diehl. *Model Predictive Control: Theory, Computation, and Design,* 2017.

Review:

MPC solves finite-time OCPs in a receding horizon fashion

(1) For computational reasons

(2) To incorporate latest informations



$$\min_{u_{t|t},\ldots,u_{t+N-1|t}} l_T\left(x_{t+N|t}\right) + \sum_{k=0}^{N-1} l\left(x_{t+k|t}, u_{t+k|t}\right)$$

$$\text{s.t} \quad x_{t+k+1|t} = Ax_{t+k|t} + Bu_{t+k|t}, k = 0,\ldots,N-1$$

$$x_{t+k|t} \in X, \quad k = 0,\ldots,N-1$$

$$u_{t+k|t} \in U, \quad k = 0,\ldots,N-1$$

$$x_{t+N|t} \in X_f$$

$$x_{t|t} = x(t)$$

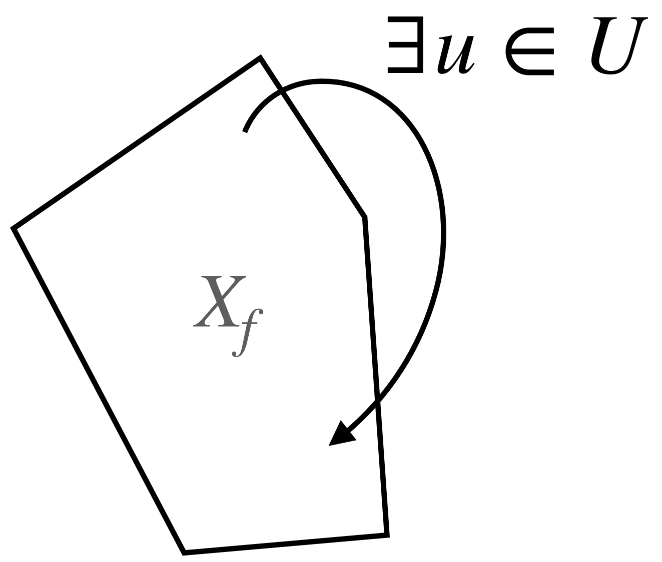
Main issues

(1) Ensure persistent feasibility

(2) Stability

How to approach (1)?

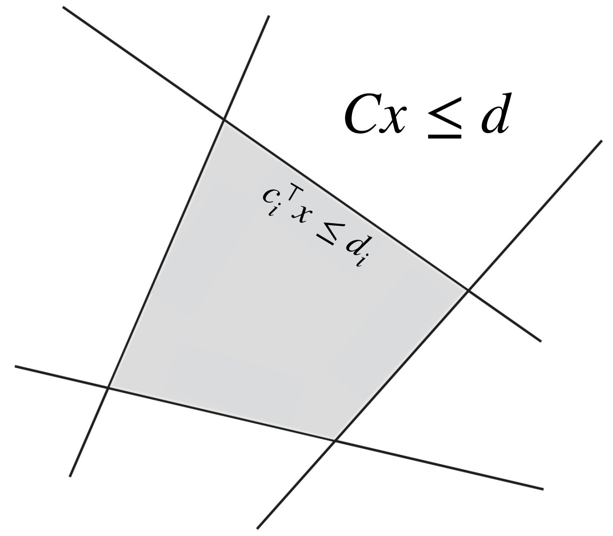
Define the terminal constraint set X_f to be control invariant (as large as possible)



Mathematically, we focused on LTI systems

Goal: design MPC controller so that feasibility for all future times is guaranteed

Approach: leverage tools from *invariant set theory*



$$J_0^*(x(t)) = \min_{u_0,\ldots,u_{N-1}} l_T\left(x_N\right) + \sum_{k=0}^{N-1} l\left(x_k, u_k\right)$$

$$\text{s.t} \quad x_{k+1} = Ax_k + Bu_k, \quad k = 0,\ldots,N-$$

$$x_k \in X, \quad k = 0,\ldots,N-1$$

$$u_k \in U, \quad k = 0,\ldots,N-1$$

$$x_N \in X_f$$

$$x_0 = x(t)$$

Feasibility theorem:

If set X_f is a control invariant set for system

$$x(t+1) = Ax(t) + Bu(t), \quad x(t) \in X, \quad u(t) \in U, \quad t \geq 0$$

, then the MPC law is persistently feasible

Outline of the next two lectures

MPC: Basic setting and key ideas

Main design choices:
- Persistent feasibility
- Stability

Implementation aspects of MPC

Further reading:
- F. Borrelli, A. Bemporad, M. Morari. *Predictive Control for Linear and Hybrid Systems*, 2017.
- J. B. Rawlings, D. Q. Mayne, M. M. Diehl. *Model Predictive Control: Theory, Computation, and Design,* 2017.

Stability of MPC

- Persistent feasibility does not guarantee that the closed-loop trajectories converge towards the desired equilibrium point

- One of the most popular approaches to guarantee persistent feasibility and stability of the MPC law makes use of a control invariant terminal set X_f for feasibility, and of a terminal cost $l_T(\cdot)$ for stability

- To prove stability, we leverage the tool of Lyapunov stability theory

Theorem (Lyapunov's direct method)

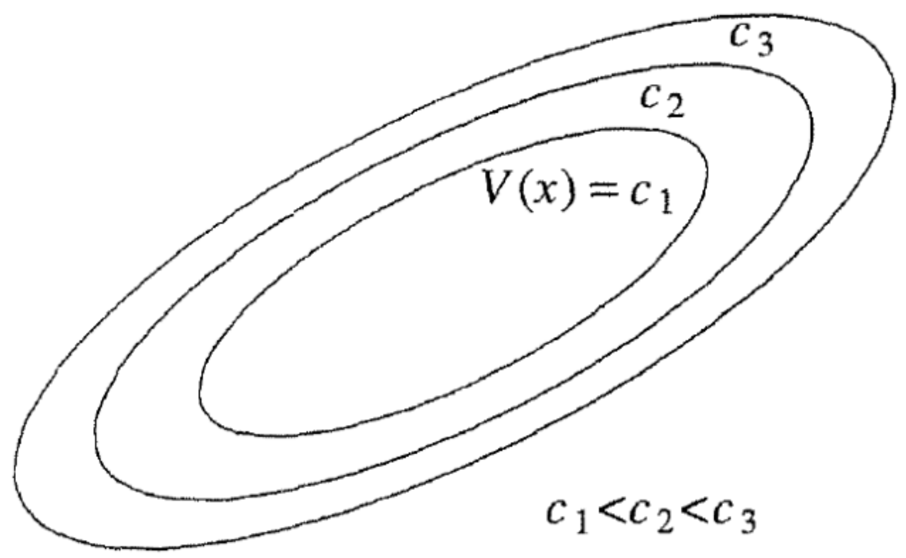
Consider $\dot{x} = f(x)$ where f is locally Lipschitz and $f(0) = 0$. Suppose there exists $V \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ such that

- *V is positive-definite, i.e., $V(x) \geq 0$ and $V(x) = 0 \iff x = 0$,*
- *$\dot{V}$ is negative-definite, i.e., $\nabla V(x)^\mathsf{T} f(x) \leq 0$ and $\nabla V(x)^\mathsf{T} f(x) = 0 \iff x = 0$.*

Then $\bar{x} = 0$ is locally asymptotically stable. If in addition

- *V is radially unbounded, i.e., $V(x) \to \infty$ as $\|x\| \to \infty$,*

then $\bar{x} = 0$ is globally asymptotically stable.



If the "energy" $V(x)$ is decreasing everywhere along trajectories, then $V(x) \to 0$ and thus $x \to 0$.

Lyapunov Stability Theorem (in discrete time)

Lyapunov Theorem:

- Consider the equilibrium point $x = 0$ for the autonomous system $\{x_{k+1} = f(x_k)\}$ (with $f(\mathbf{0}) = 0$).
- Let $\Omega \subset R^n$ be a closed, bounded, positively invariant set containing the origin.
- Let $V : \mathbb{R}^n \rightarrow \mathbb{R}$ be a function, continuous at the origin, such that

$$V(\mathbf{0}) = 0 \text{ and } V(x) > 0 \quad \forall x \in \Omega \backslash \{\mathbf{0}\}$$
$$V\left(x_{k+1}\right) - V\left(x_k\right) < 0 \quad \forall x_k \in \Omega \backslash \{\mathbf{0}\}$$

→ then $x = 0$ is asymptotically stable in Ω

- The idea is to show that with appropriate choices of X_f and $l_T(\cdot)$, J_0^* is a Lyapunov function for the closed-loop system

MPC Stability Theorem

$$J_0^*(x(t)) = \min_{u_0,\ldots,u_{N-1}} l_T\left(x_N\right) + \sum_{k=0}^{N-1} l\left(x_k, u_k\right)$$

where $\quad l_T(x) = x^\top P x, \quad l(x,u) = x^\top Q x + u^\top R u$

MPC Stability Theorem (for quadratic cost):

Assume:

A0: $Q = Q^T \succ 0, \quad R = R^T \succ 0, \quad P \succ 0$

A1: Sets X, X_f, and U contain the origin in their interior and are closed

A2: $X_f \subseteq X$ is control invariant and bounded

A3: $\min_{u \in U, Ax+Bu \in X_f} (-l_T(x) + l(x,u) + l_T(Ax+Bu)) \leq 0, \forall x \in X_f$

Then, the origin of the closed-loop system is asymptotically stable with domain of attraction X_0

MPC Stability Theorem

$$J_0^*(x(t)) = \min_{u_0,\ldots,u_{N-1}} x_N^\top P x_N + \sum_{k=0}^{N-1} x_k^\top Q x_k + u_k^\top R u_k$$

$$\begin{aligned}
\text{s.t} \quad & x_{k+1} = Ax_k + Bu_k, \quad k = 0,\ldots,N-1 \\
& x_k \in X, \quad k = 0,\ldots,N-1 \\
& u_k \in U, \quad k = 0,\ldots,N-1 \\
& x_N \in X_f \\
& x_0 = x(t)
\end{aligned}$$

Proof:

1. Note that, by assumption A2, persistent feasibility is guaranteed for any P, Q, R
2. We want to show that J_0^* is a Lyapunov function for the closed-loop system $x(t+1) = f_{\mathrm{cl}}(x(t))$, with respect to the equilibrium $f_{\mathrm{cl}}(\mathbf{0}) = 0$

   (the origin is indeed an equilibrium as $\mathbf{0} \in X, \mathbf{0} \in U$, and the cost is positive for any non-zero control sequence)
3. X_0 is bounded and closed

   (follows from assumption on X_f)
4. $J_0^*(\mathbf{0}) = 0$ (value is nonnegative by construction, and 0 is achievable) ✓
5. $J_0^*(x) > 0$ for all $x \in X_0 \backslash \{0\}$ ✓✓
6. Next, we check for the decaying property (i.e., $J_0^*(x(k+1)) - J_0^*(x(k)) < 0$)

$f_{\mathrm{cl}}(x(t)) \rightarrow x_{k+1} = Ax_k + B\pi(x_k),$ where $\pi() =$ MPC Optimization Problem

MPC Stability Theorem

Proof:

7. Since the setup is time-invariant, we can study the decay property between $t = 0$ and $t = 1$
   - Let $x(0) \in X_0$, let $U_0^{[0]} = \left[u_0^{[0]}, u_1^{[0]}, \ldots, u_{N-1}^{[0]}\right]$ be the optimal control sequence, and let $\left[x(0), x_1^{[0]}, \ldots, x_N^{[0]}\right]$ be the corresponding (predicted) trajectory
   - After applying $u_0^{[0]}$, one obtains $x(1) = Ax(0) + Bu_0^{[0]}$
   - Consider the sequence of controls $\left[u_1^{[0]}, u_2^{[0]}, \ldots, u_{N-1}^{[0]}, v\right]$, where $v \in U$, and the corresponding state trajectory is $\left[x(1), x_2^{[0]}, \ldots, x_N^{[0]}, Ax_N^{[0]} + Bv\right]$
8. Since $x_N^{[0]} \in X_f$ (by terminal constraint), and since X_f is control invariant,

$$\exists \bar{v} \in U, \text{ such that } Ax_N^{[0]} + B\bar{v} \in X_f$$

9. With such a choice of $\bar{v}$, the sequence $\left[u_1^{[0]}, u_2^{[0]}, \ldots, u_{N-1}^{[0]}, \bar{v}\right]$ is feasible for the MPC optimization problem at time $t = 1$

MPC Stability Theorem

A3: $\min_{u \in U, Ax+Bu \in X_f} (-l_T(x) + l(x,u) + l_T(Ax+Bu)) \le 0, \forall x \in X_f$

Proof:

10. Since this sequence is not necessarily optimal

$$J_0^*(x(1)) \le l_T\left(Ax_N^{[0]} + B\bar{v}\right) + \sum_{k=1}^{N-1} l\left(x_k^{[0]}, u_k^{[0]}\right) + l\left(x_N^{[0]}, \bar{v}\right)$$

$$+ l_T\left(x_N^{[0]}\right) - l_T\left(x_N^{[0]}\right) + l\left(x(0), u_0^{[0]}\right) - l\left(x(0), u_0^{[0]}\right)$$

11. Equivalently,

$$J_0^*(x(1)) \le l_T\left(Ax_N^{[0]} + B\bar{v}\right) + J_0^*(x(0)) + l\left(x_N^{[0]}, \bar{v}\right) - l_T\left(x_N^{[0]}\right) - l\left(x(0), u_0^{[0]}\right)$$

- Since $x_N^{[0]} \in X_f$, by assumption A3, we can select $\bar{v}$ such that

$$J_0^*(x(1)) \le J_0^*(x(0)) - l\left(x(0), u_0^{[0]}\right)$$

- Moreover, since $l\left(x(0), u_0^{[0]}\right) > 0$ for all $x(0) \in X_0 \backslash \{0\}$, we can write

$$J_0^*(x(1)) - J_0^*(x(0)) < 0 \quad \checkmark\checkmark\checkmark$$

MPC Stability Theorem

Note:

- The last step in the proof is to prove continuity; details are omitted and can be found in *Borrelli, Bemporad, Morari, 2017*
- A2 (i.e., $X_f \subseteq X$ is control invariant and bounded) is used to guarantee persistent feasibility; this assumption can be replaced with an assumption on the horizon N

How to choose X_f and l_T?

In this and the previous lecture, we derived two general criterial for choosing the terminal constraint and cost of our short-term problem. Namely:

1) X_f control invariant (from persistent feasibility theorem)

2) l_T satisfies A3

$$\min_{u \in U, Ax+Bu \in X_f} \left(-l_T(x) + l(x,u) + l_T(Ax+Bu)\right) \leq 0, \forall x \in X_f$$

(from stability theorem)

Let us consider two cases where we describe two specific choices of X_f and l_T

How to choose X_f and ~~l_T~~, P? (Case 1)

Consider

1. The system

$$\mathbf{x}(t+1) = A\mathbf{x}(t) + B\mathbf{u}(t), \quad \mathbf{x}(t) \in \mathbb{R}^n, \quad \mathbf{u}(t) \in \mathbb{R}^m$$

s.t. $\mathbf{x}(t) \in X, \quad \mathbf{u}(t) \in U, \quad t \geq 0$

2. The RHC control law

$$J_0^*(x(t)) = \min_{u_0,\ldots,u_{N-1}} l_T\left(x_N\right) + \sum_{k=0}^{N-1} l\left(x_k, u_k\right)$$

s.t
$$x_{k+1} = Ax_k + Bu_k, \quad k = 0,\ldots,N-1$$
$$x_k \in X, \quad k = 0,\ldots,N-1$$
$$u_k \in U, \quad k = 0,\ldots,N-1$$
$$x_N \in X_f$$
$$x_0 = x(t)$$

$$U_0^*(x(t)) = \left\{u_0^*, \ldots, u_{N-1}^*\right\} \qquad \pi(\mathbf{x}(t)) := \mathbf{u}_0^*$$

3. Cost function $J_0\left(x(0)\right) = x_N^\top P x_N + \sum_{k=0}^{N-1} x_k^\top Q x_k + u_k^\top R u_k$

Set:

- X_f as the maximally positive invariant set for the closed-loop system $x(t+1) = \left(A + BF_\infty\right)x(t)$
  - (With constraints $x(t) \in X$, and $F_\infty x(t) \in U$)
  - Where F_∞ is the optimal gain for the infinite-horizon LQR controller
- P as the solution P_∞ to the discrete-time Riccati equation, i.e., the value function via LQR

How to choose X_f and ~~l_T~~, P? (Case 2)

Consider the same setting as before, where A is asymptotically stable

Set:

- X_f as the maximally positive invariant set for the closed-loop system $x(t+1) = Ax(t)$
  - (With constraints $x(t) \in X$)
- X_f is a control invariant set for the system $x(t+1) = Ax(t) + Bu(t)$, as $u = 0$ is a feasible control
- As for stability, $u = 0$ is feasible and $Ax \in X_f$ if $x \in X_f$, thus assumption A3 becomes

$$-x^T P x + x^T Q x + x^T A^T P A x \leq 0, \text{ for all } x \in X_f,$$

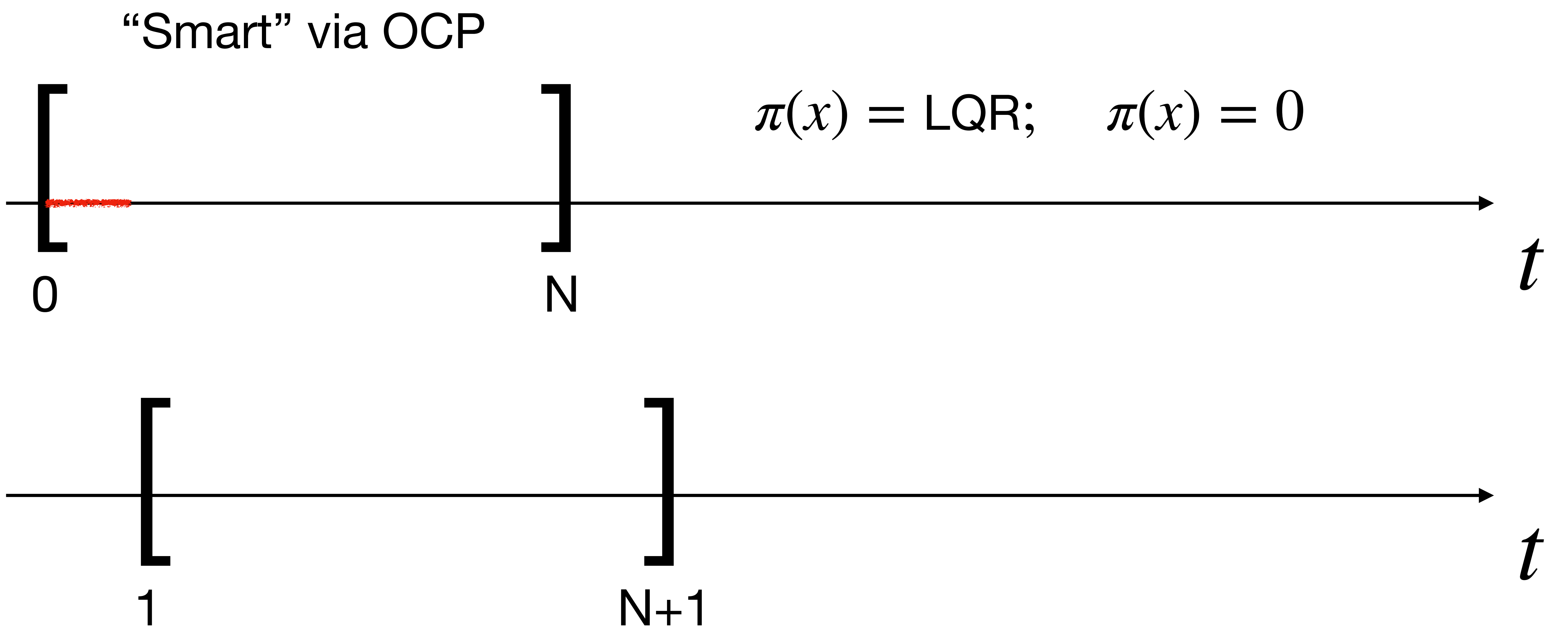
which, due to the fact that A is asymptotically stable, it is satisfied as an equality if we choose P as a solution of the corresponding Lyapunov equation

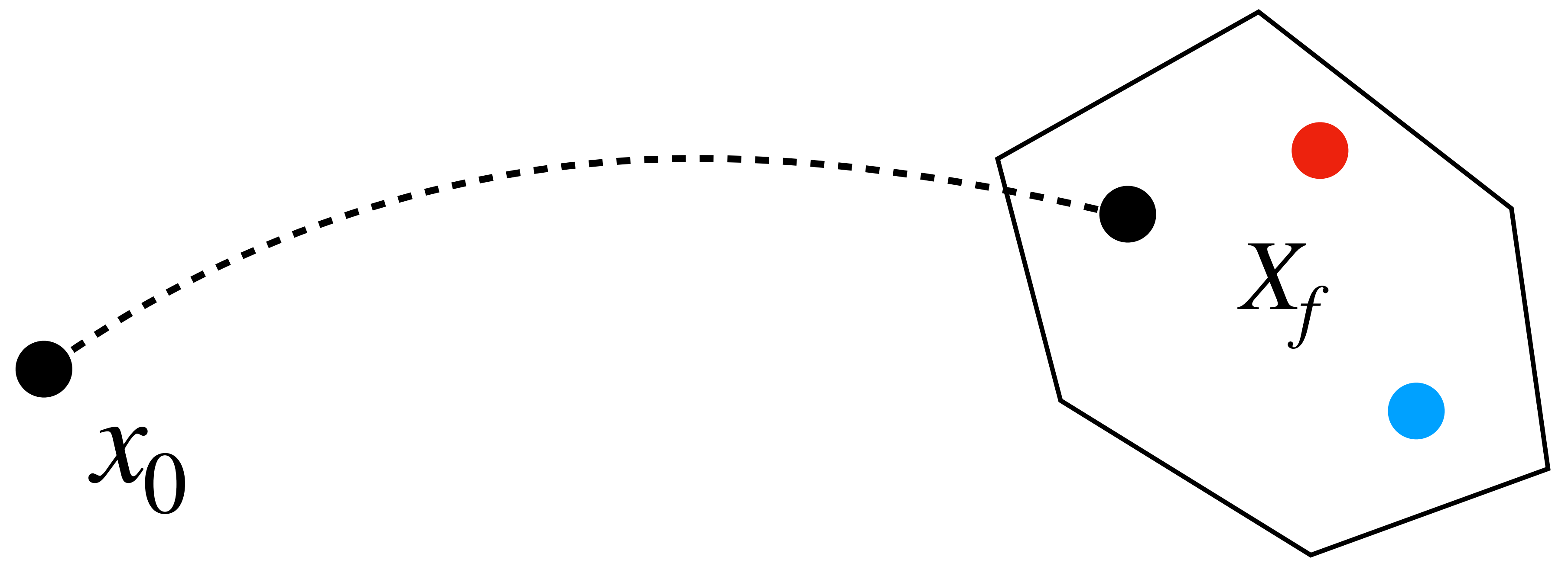
$$\exists P \succ 0 \mid -P + Q + A^T P A = 0$$

Intuition

Note: both cases as presented are just (suboptimal) choices!

- We care about a (potentially) infinite-horizon problem and design a strategy to solve (in a receding horizon fashion) OCP for the first N steps
- We discussed how X_f and l_T are key design choices
  - X_f as "a set of states where we are safe"
  - l_T to "guide performance by approximating the long-horizon problem" → cost-to-go!
- In other words, use optimization over the first N steps to act "smart"
- Approximate the long-horizon cost under some policy e.g., LQR





Tuning and practical use

- At present there is no other technique than MPC to design controllers for general large linear multivariable systems with input and output constraints with a stability guarantee

- Design approach (for squared 2-norm cost):
  - Choose horizon length N and the control invariant target set Xf
  - Control invariant target set Xf should be as large as possible for performance
  - Choose the parameters Q and R freely to affect the control performance
  - Adjust P as per the stability theorem
  - Useful toolbox (MATLAB): https://www.mpt3.org/

- In practice, sometimes choosing a good terminal cost is enough (i.e., don't need to enforce a terminal control invariant condition), though you may be sacrificing guarantees

Explicit MPC

- In some cases, the MPC law can be pre-computed → no need for online optimization
- Important case: constrained LQR

$$J_0^*(x(t)) = \min_{u_0,\ldots,u_{N-1}} x_N^\top P x_N + \sum_{k=0}^{N-1} x_k^\top Q x_k + u_k^\top R u_k$$

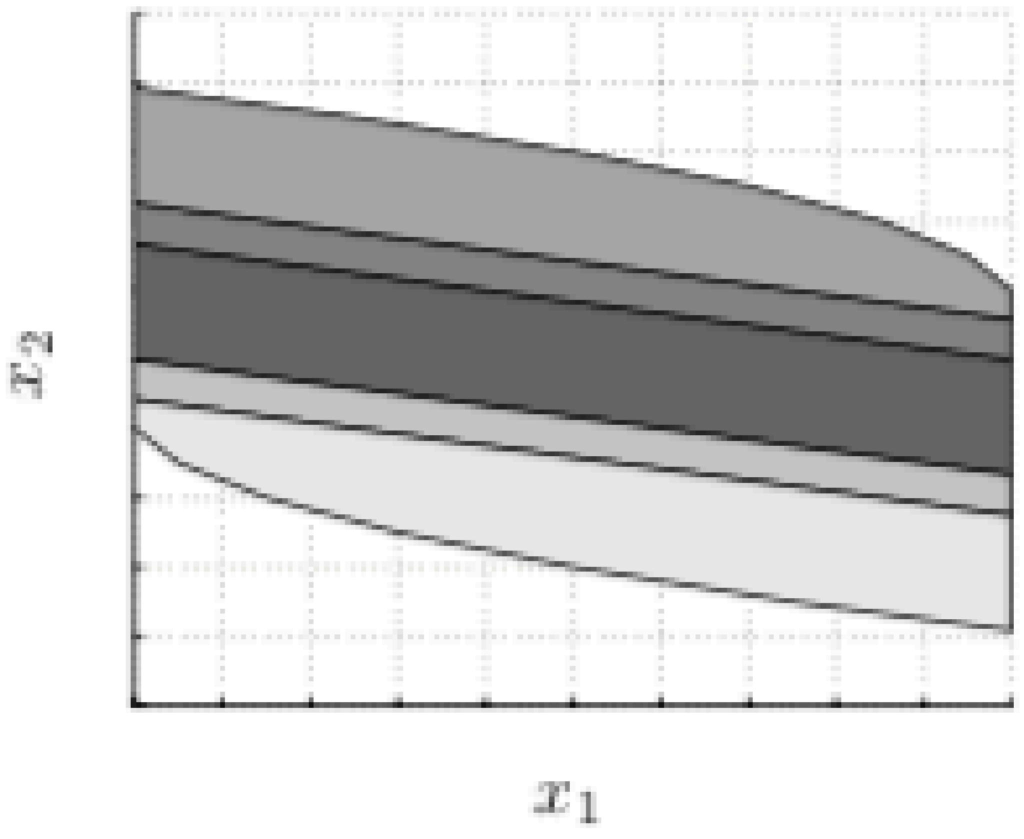
$$\begin{aligned} \text{s.t} \quad & x_{k+1} = Ax_k + Bu_k, \quad k = 0,\ldots,N-1 \\ & x_k \in X, \quad k = 0,\ldots,N-1 \\ & u_k \in U, \quad k = 0,\ldots,N-1 \\ & x_N \in X_f \\ & x_0 = x(t) \end{aligned}$$

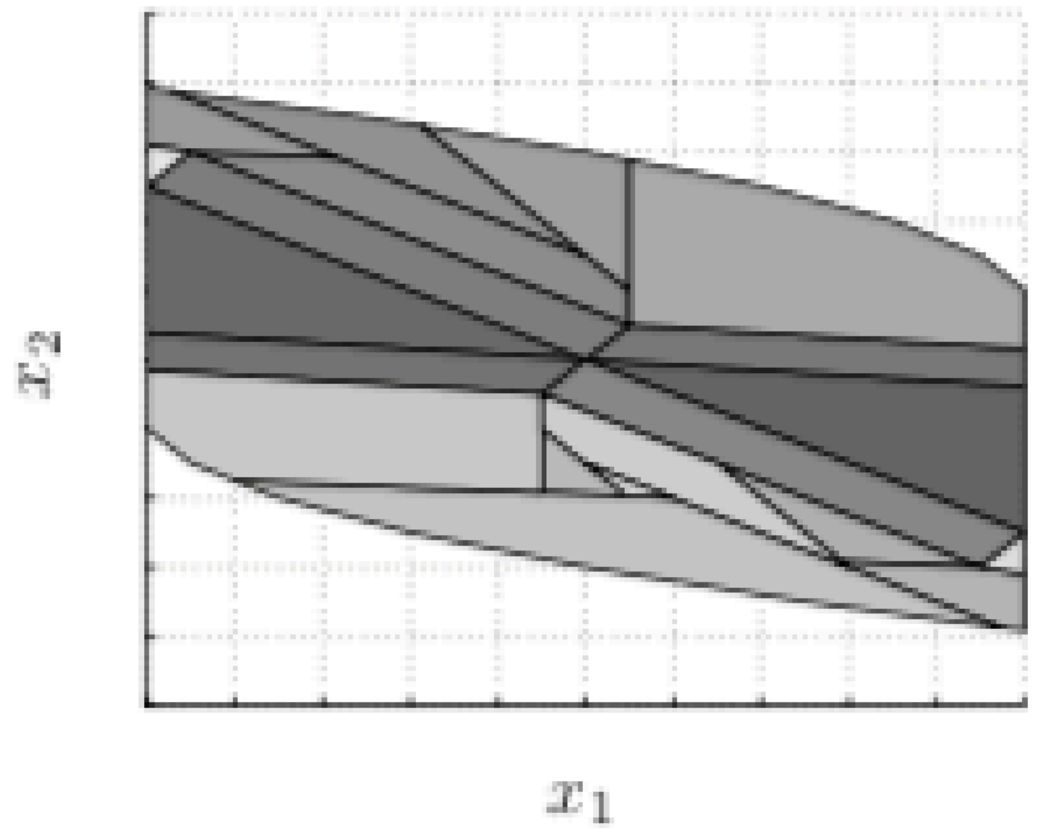
Explicit MPC

- The solution to the constrained LQR problem is a control which is a continuous piecewise affine function on polyhedral partition of the state space X, that is $u_k^* = \pi_k(x_k)$ where

$$\pi_k(x) = F_k^j x + g_k^j \text{ if } H_k^j x \leq K_k^j, j = 1, \ldots, N_k^r$$

- Thus, online, one has to locate in which cell of the polyhedral partition the state x lies, and then one obtains the optimal control via a look-up table query





MPC for reference tracking

- Usual cost

$$\sum_{k=0}^{N-1} x_k^T Q x_k + u_k^T R u_k$$

does not work, as in steady state control does not need to be zero

- δu-formulation: reason in terms of control changes

$$u_k = u_{k-1} + \delta u_k$$

MPC for reference tracking

- The MPC problem is readily modified to

$$J_0^*(x(t)) = \min_{\delta u_0,\ldots,\delta u_{N-1}} \sum_k \| y_k - r_k \|_Q^2 + \| \delta u_k \|_R^2$$

subject to $x_{k+1} = Ax_k + Bu_k, \quad k = 0,\ldots,N-1$

$$y_k = Cx_k, \quad k = 0,\ldots,N-1$$

$$x_k \in X, \quad u_k \in U, \quad k = 0,\ldots,N-1$$

$$x_N \in X_f$$

$$x_k = u_{k-1} + \delta u_k, \quad k = 0,\ldots,N-1$$

$$x_0 = x(t), \quad u_{-1} = u(t-1)$$

- The control input is then

$$u(t) = \delta u_0^* + u(t-1)$$

Next time

- Robust MPC
- Adaptive MPC